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Logical Quantization of Differential Geometry

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In a previous paper we saw that Grothendieck's functorial approach to algebraic geometry and algebraic groups in particular is in consonance with our framework of logical quantization. This paper, as a sequel, consolidates the consonance between functorial geometry and logical quantization by demonstrating that Moerdijk and Reyes' functorial approach to differential geometry can be adequately poised within logical quantization.

0. INTRODUCTION

There are at least two principal approaches to algebraic geometry and algebraic groups in particular, namely, geometric and functorial ones, to both of which A. Grothendieck has made decisive contributions. The former culminates in his celebrated scheme theory, which any not-too-elementary textbook on modern algebraic geometry is obliged to deal with in some way or other. See, e.g., Hartshorne (1977), which is considered the standard textbook on algebraic geometry of our day. For the latter stream in algebraic geometry the reader is referred to the worthwhile book by Demazure and Gabriel (1980).

We know well that infinitesimals were active in the realm of analysis during the days of I. Newton and G. W. F. Leibniz and that they were rampant throughout the works of such pioneers in the arena of modern geometry as E. Cartan, S. Lie, and B. Riemann. Although the so-called $\epsilon - \delta$ arguments made mathematics rigorous by eradicating infinitesimals relentlessly, many vivid intuitions of those good old days were completely lost in the labyrinth of logic.

Nowadays infinitesimals are coming back in mathematics through two streams. One is nonstandard analysis, which is an application of model

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theory to analysis. Model theory can provide many nonstandard models of mathematical theories, in particular, models of analysis in which infinitesimals are consistently alive. For nonstandard analysis the reader is referred, e.g., to Robinson (1966) and Stroyan and Luxemberg (1976).

The renaissance of infinitesimals can be seen also in synthetic differential geometry, in which infinitesimals are redeemed by weaking the underlying logic from classical to intuitionistic, so that every argument and each construction should be constructive. In contrast to nonstandard analysis, synthetic differential geometry employs nilpotent infinitesimals besides invertible ones. For synthetic differential geometry the reader is referred to Kock's (1981) Bible, and for its functorial semantics he or she is referred to Moerdijk and Reyes (1991), in which, among other things, a highly intuitive proof of the so-called Gauss–Bonnet theorem in dimension 2 is given simply by adding infinitesimal angles in various ways.

In a previous paper (Nishimura, 1995c) we showed that Grothendieck's functorial approach to algebraic geometry matches well with our developing theory of logical quantization. This paper is devoted, as a sequel to the above one, to showing that the functorial semantics of differential geometry falls neatly into place within our favorite framework of logical quantization, enhancing our tenet that functorial semantics is concordant with logical quantization. Moerdijk and Reyes (1991) deal with four models eligible for synthetic differential geometry (called smooth toposes), all of which are examples of Grothendieck toposes. Therefore this paper is concerned almost exclusively with the logical quantization of Grothendieck toposes, only touching upon the smooth Zariski topos among their four smooth toposes, leaving truly geometric considerations to subsequent papers (e.g., Nishimura, n.d.-a).

The organization of this paper goes as follows: Section 1 is devoted to Booleanization of algebraic theories a là Lawvere and relations between two such Booleanizations over possibly different complete Boolean algebras. Section 2 deals with Booleanization of Grothendieck toposes and relations between two such Booleanizations over possibly different complete Boolean algebras, rectifying the hasty treatments of Theorem 2.6 and some others of our previous paper (Nishimura, 1995c). Algebraic theories and Grothendieck toposes are logically quantized in Section 3. A miscellany of examples is given as an appendix, and the reader is referred to them upon occasion. He or she should be familiar with all of them before reading Section 3.

We assume the reader to be familiar with our previous paper (Nishimura, 1995c), though we are not necessarily faithful to its notation and terminology. The rest of this section is devoted to fixing notation and providing some preliminaries.

0.1. Universes of Small Sets

To dodge the famous paradoxes of set theory or to paper them over, the usage of a *universe* **U**, which is by definition a set of sets closed under all reasonable set-theoretic operations, is a common practice in category theory. For the exact definition of a universe the reader is referred to MacLane (1971, Chapter I, §6), Schubert (1972, §3.2), and Borceux (1994, Vol. 1, §1.1). In this paper we assume that there are two universes $\mathbf{V}_0$ and $\mathbf{V}_1$ with $\mathbf{V}_0 \in \mathbf{V}_1$. A set belonging to $\mathbf{V}_i$ is called *small*$_i$ ($i = 0, 1$). The adjective "small$_i$" is applied to structures whose underlying sets are small$_i$. The category of small$_i$ sets and small$_i$ functions is denoted by $\mathbf{Ens}_i$.

0.2. Boolean Locales

The category of small$_0$ complete Boolean algebras and their complete Boolean homomorphisms is denoted by **Bool**. Its dual category is denoted by **BLoc**. The objects of **BLoc** are called *Boolean locales* and are denoted by X, Y, ... The morphisms of **BLoc** are denoted by f, g, ... If a Boolean locale X is to be put down as an object of **Bool**, it is denoted by $\mathscr{P}$(X) for emphasis, though X and $\mathscr{P}$(X) denote the same entity. The morphism of **Bool** corresponding to a morphism f: X → Y of **BLoc** is denoted by $\mathscr{P}^*$(f), while the right-adjoint of $\mathscr{P}^*(\mathrm{f}): \mathscr{P}(\mathrm{Y}) \to \mathscr{P}(\mathrm{X})$ whose existence is guaranteed by Theorem 2.1 of Nishimura (1993b) is denoted by $\mathscr{P}_*$(f). A *manual of Boolean locales* is a small$_0$ subcategory of **BLoc** satisfying certain mild constraints, as was the case in our previous paper (Nishimura, 1995c).

0.3. X-Sets and X-Sets

Let X be a Boolean locale, which shall be fixed throughout this subsection. We will often write **B** for $\mathscr{P}$(X). An X-set is a pair $(U, [\![\cdot = \cdot]\!]_{\mathrm{X}}^U)$ of a set U and a function $[\![\cdot = \cdot]\!]_{\mathrm{X}}^U: U \times U \to \mathbf{B}$ abiding by the following conditions:

(0.3.1) $[\![x = y]\!]_{\mathrm{X}}^U = [\![y = x]\!]_{\mathrm{X}}^U$
(0.3.2) $[\![x = y]\!]_{\mathrm{X}}^U \wedge [\![y = z]\!]_{\mathrm{X}}^U \leq [\![x = z]\!]_{\mathrm{X}}^U$

for all $x, y, z \in U$. We will often write $[\![x = y]\!]_{\mathrm{X}}$, $[\![x = y]\!]^U$, or simply $[\![x = y]\!]$ for $[\![x = y]\!]_{\mathrm{X}}^U$, unless confusion may arise. We will often write $\mathrm{E}_{\mathrm{X}}^U x$, $\mathrm{E}^U x$, $\mathrm{E}_{\mathrm{X}} x$, or $\mathrm{E}x$ for $[\![x = x]\!]$. An X-set $(U, [\![\cdot = \cdot]\!])$ is often represented simply by its underlying set U. Given X-sets $(U, [\![\cdot = \cdot]\!]^U)$ and $(V, [\![\cdot = \cdot]\!]^V)$, we write $(U, [\![\cdot = \cdot]\!]^U) \times_{\mathrm{X}} (V, [\![\cdot = \cdot]\!]^V)$ for the X-set $(U \times_{\mathrm{X}} V, [\![\cdot = \cdot]\!]^{U \times_{\mathrm{X}} V})$, where

(0.3.3) $U \times_{\mathrm{X}} V = \{(x, y) \in U \times V \mid \mathrm{E}^U x = \mathrm{E}^V y\}$
(0.3.4) $[\![(x, y) = (x', y')]\!]^{U \times_{\mathrm{X}} V} = [\![x = x']\!]^U \wedge [\![y = y']\!]^V$ for all (x, y), $(x', y') \in U \times_{\mathrm{X}} V$

To make the set of all small$_i$ X-$\underline{\text{sets}}$ a category $\underline{\mathbf{Bens}}_i(\mathrm{X})$ ($i = 0, 1$), we need to define a morphism from a small$_i$ X-$\underline{\text{set}}$ U to a small X-$\underline{\text{set}}$ V, which is to be a function δ: $U \times V \to \mathbf{B}$ abiding by the following conditions:

$$(0.3.5) \quad [\![x = x']\!]^U \wedge \delta(x, y) \leq \delta(x', y)$$
$$(0.3.6) \quad \delta(x, y) \wedge [\![y = y']\!]^V \leq \delta(x, y')$$
$$(0.3.7) \quad \delta(x, y) \wedge \delta(x, y') \leq [\![y = y']\!]^V$$
$$(0.3.8) \quad \vee_{y \in V}\delta(x, y) = \mathrm{E}x$$

for all $x, x' \in U$ and all $y, y' \in V$.

Given an X-$\underline{\text{set}}$ $(U, [\![\cdot = \cdot]\!])$, a function α: $U \to \mathbf{B}$ is called a *singleton* if it satisfies the following conditions:

$$(0.3.9) \quad \alpha(x) \wedge [\![x = y]\!] \leq \alpha(y)$$
$$(0.3.10) \quad \alpha(x) \wedge \alpha(y) \leq [\![x = y]\!]$$

for all $x, y \in U$. It is easy to see that each $x \in U$ gives rise to a singleton $\{x\}$ assigning to each $y \in U$ $[\![x = y]\!] \in \mathbf{B}$. The X-$\underline{\text{set}}$ $(U, [\![\cdot = \cdot]\!])$ is called an X-*set* if every singleton is of the form $\{x\}$ for a unique $x \in U$. We denote by $\mathbf{BEns}_i(\mathrm{X})$ the full subcategory of $\underline{\mathbf{BEns}}_i(\mathrm{X})$ whose objects are all X-sets ($i = 0, 1$). As is discussed in Goldblatt (1979, §§11.9 and 14.7), the categories $\underline{\mathbf{BEns}}_i(\mathrm{X})$ and $\mathbf{BEns}_i(\mathrm{X})$ are toposes. As we have discussed in Nishimura (1995b, Theorem 1.2), there is a geometric morphism $(\mathbf{i}_{\mathbf{BEns}_i}[\mathrm{X}], \mathbf{a}_{\mathbf{BEns}_i}[\mathrm{X}])$ from $\mathbf{BEns}_i(\mathrm{X})$ to $\underline{\mathbf{BEns}}_i(\mathrm{X})$.

Let U be a small$_1$ X-$\underline{\text{set}}$ and V a small$_1$ X-set. Then there is a natural bijection between the morphisms from U to V in $\mathbf{BEns}_1(\mathrm{X})$ and the functions f: $U \to V$ yielding the following conditions:

$$(0.3.11) \quad [\![x = y]\!]^U \leq [\![f(x) = f(y)]\!]^V$$
$$(0.3.12) \quad \mathrm{E}^V f(x) \leq \mathrm{E}^U x$$

for all $x, y \in U$. The reader is referred to Goldblatt (1979, §14.7) for the detailed construction of this well-known bijection.

Let f: $\mathrm{X}_- \to \mathrm{X}_+$ be a morphism in **BLoc**. Then the assignment

$$(U, [\![\cdot = \cdot]\!]^U) \in \mathrm{Ob}\ \underline{\mathbf{BEns}}_1(\mathrm{X}_+) \mapsto (U, \mathscr{P}^*(\mathrm{f})([\![\cdot = \cdot]\!]^U)) \in \mathrm{Ob}\ \underline{\mathbf{BEns}}_1(\mathrm{X}_-)$$

naturally induces a functor $\underline{\underline{\mathrm{f}}}^*$: $\underline{\mathbf{BEns}}_1(\mathrm{X}_+) \to \underline{\mathbf{BEns}}_1(\mathrm{X}_-)$, which in turn gives rise to functors

$$\underline{\mathrm{f}}^* = \underline{\underline{\mathrm{f}}}^* \circ \mathbf{i}_{\mathbf{BEns}_1}[\mathrm{X}_+]: \mathbf{BEns}_1(\mathrm{X}_+) \to \underline{\mathbf{BEns}}_1(\mathrm{X}_-)$$

$$\mathrm{f}^* = \mathbf{a}_{\mathbf{BEns}_1}[\mathrm{X}_-] \circ \underline{\mathrm{f}}^*: \mathbf{BEns}_1(\mathrm{X}_+) \to \mathbf{BEns}_1(\mathrm{X}_-)$$

On the other hand, the assignment

$$(V, [\![\cdot = \cdot]\!]^V) \in \mathrm{Ob}\ \mathbf{BEns}_1(\mathrm{X}_-)$$
$$\mapsto \mathbf{a}_{\mathbf{BEns}_1}[\mathrm{X}_+](V, \mathscr{P}_*(\mathrm{f})([\![\cdot = \cdot]\!]^V)) \in \mathrm{Ob}\ \mathbf{BEns}_1(\mathrm{X}_+)$$

naturally induces a functor f_*: $\mathbf{BEns}_1(X_-) \to \mathbf{BEns}_1(X_+)$. As we discussed in Nishimura (1993b, §2), the pair (f_*, f^*) forms a geometric morphism from $\mathbf{BEns}_1(X_-)$ to $\mathbf{BEns}_1(X_+)$, i.e., $f^* \dashv f_*$ and f^* is left-exact. Since the geometric morphism (f_*, f^*): $\mathbf{BEns}_1(X_-) \to \mathbf{BEns}_1(X_+)$ corresponds to the morphism f: $X_- \to X_+$ in **BLoc** under Theorem 2.6 of Nishimura (1993b) and f is open by Theorem 2.13 of Nishimura (1993b), the geometric morphism (f_*, f^*) is essential due to Exercise 2.13.8 of Borceux (1994, Vol. 3) in the sense that f^* has a left-adjoint $f_!$: $\mathbf{BEns}_1(X_-) \to \mathbf{BEns}_1(X_+)$. In particular, the functor f^*: $\mathbf{BEns}_1(X_+) \to \mathbf{BEns}_1(X_-)$ preserves not only arbitrary colimits, but also arbitrary limits by dint of Theorem 1 of MacLane (1971, Chapter V, §5).

0.4. Two Transfer Principles

Let X be a Boolean locale with $\mathbf{B} = \mathscr{P}(X)$. As we have discussed in Nishimura (1993b), the topos $\mathbf{BEns}_1(X)$ is equivalent to the category of sets and functions within the Scott-Solovay universe $\mathbf{V}_1^{(\mathbf{B})}$. As Jech (1978, Theorem 43) and others have discussed, the universe $\mathbf{V}_1^{(\mathbf{B})}$ enjoys ZFC (Zermelo–Fraenkel set theory with the axiom of choice), which is the core principle of Boolean mathematics. For Boolean mathematics, the reader is referred, e.g., to Nishimura (1984, 1991, 1992, 1993a), Ozawa (1983, 1984, 1985), Smith (1984), and the Bible of Boolean mathematics, Takeuti (1978). Since every branch of mathematics, ranging from algebraic geometry to functional analysis, is in principle to be developed within ZFC, the Scott-Solovay universe $\mathbf{V}_1^{(\mathbf{B})}$ and therefore its equivalent $\mathbf{BEns}_1(X)$ enjoy all classical mathematics (=mathematics based on classical logic). This transfer principle from standard mathematics to Boolean mathematics is designated the *Zermelo–Fraenkel transfer principle* (ZFTP). The application of the transfer principle is usually called *Booleanization.*

Let f: $X_- \to X_+$ be a morphism of **BLoc**. Due to Theorem 2.13 of Nishimura (1993b), f is open, so that the geometric morphism (f_*, f^*): $\mathbf{BEns}_1(X_-) \to \mathbf{BEns}_1(X_+)$ is also open by Proposition 2 of MacLane and Moerdijk (1992, Chapter IX, §7). This implies that every first-order property holding in a (many-sorted) first-order structure $\mathscr{A}$ in $\mathbf{BEns}_1(X_+)$ persists in the derived first-order structure $f^*\mathscr{A}$ in $\mathbf{BEns}_1(X_-)$, as is claimed in Corollary 4 of MacLane and Moerdijk (1992, Chapter X, §3). This transfer principle is designated the *first-order transfer principle* (FOTP).

0.5. X-Categories

Let X be a Boolean locale. The interpretation of a category within the topos $\mathbf{BEns}_i(X)$ gives rise to the notion of a small_i X-category ($i = 0, 1$), as discussed in Nishimura (1995c, §1). By way of example, the totality of $\mathbf{BEns}_i(X_p)$'s $[p \in \mathscr{P}(X)]$ lumps together to form an X-category $\mathscr{BEns}_i(X)$, as was dealt with in Nishimura (1995c, Example 1.1). The notion of a functor can be interpreted within the topos $\mathbf{BEns}_i(X)$ to yield the notion of a (small$_i$) X-functor of small$_i$ X-categories. Given a small$_1$ X-category $\mathscr{C}$ and $x, y \in$ Ob $\mathscr{C}$, the set

$$\{f | f: x\lceil p \to y\lceil p \in \text{Mor}\,\mathscr{C} \text{ for some } p \in \mathscr{P}(X) \text{ with } p \leq Ex \wedge Ey\}$$

can be regarded both as an X_{Ex}-set $\mathscr{C}_X(x, y)$ and as an X_{Ey}-set $\mathscr{C}^X(x, y)$. The assignment to each $y \in$ Ob $\mathscr{C}$ of the X_{Ey}-functor $\mathscr{C}_X(?, y)$: $\mathscr{C}\lceil Ey \to \mathscr{BE}ns_1(X_{Ey})$ naturally induces the covariant Yoneda embedding $\mathbf{y}_*$ of $\mathscr{C}$ into $\mathscr{BE}ns_1(X)$, while the assignment to each $x \in$ Ob $\mathscr{C}$ of the X_{Ex}-functor $\mathscr{C}^X(x, ?)$: $\mathscr{C}\lceil Ex \to \mathscr{BE}ns_1(X_{Ex})$ naturally induces the contravariant Yoneda embedding $\mathbf{y}^*$ of $\mathscr{C}$ into $\mathscr{BE}ns_1(X)$.

Let $\mathfrak{f}$: $X_- \to X_+$ be a morphism of Boolean locales. The notion of an X-functor was generalized in our previous paper (Nishimura, 1995c, §2) to that of an $\mathfrak{f}$-functor from a small$_1$ X_+-category $\mathscr{C}_+$ to a small$_1$ X_--category $\mathscr{C}_-$. By way of example, the functors $\mathfrak{f}_p^*$: $\mathbf{BEns}_1((X_+)_p) \to \mathbf{BEns}_1((X_-)_{\mathscr{P}(\mathfrak{f})(p)})$ for all $p \in \mathscr{P}(X_+)$ lump together to form an $\mathfrak{f}$-functor $\mathfrak{f}^*_{\mathscr{BE}ns_1}$: $\mathscr{BE}ns_1(X_+) \to \mathscr{BE}ns_1(X_-)$, where $\mathfrak{f}_p$ denotes the morphism of Boolean locales from $(X_-)_{\mathscr{P}(\mathfrak{f})(p)}$ to $(X_+)_p$ naturally induced by $\mathfrak{f}$. The $\mathfrak{f}$-functor $\mathfrak{f}^*_{\mathscr{BE}ns_1}$, naturally induces such $\mathfrak{f}$-functors as $\mathfrak{f}^*_{\mathscr{BR}ng}$, which was discussed amply in our previous paper (Nishimura, 1995c). Unless confusion may occur, the superscripts in such notations as $\mathfrak{f}^*_{\mathscr{BE}ns_1}$ and $\mathfrak{f}^*_{\mathscr{BR}ng}$ are often omitted, so that the notation $\mathfrak{f}^*$ enjoys a bit of polysemy.

0.6. Miscellaneous Remarks

We denote by **R** and **Z** the set of real numbers and that of natural numbers (beginning with 0), respectively. A ring always means a commutative ring with unity, and so homomorphisms of rings are required to preserve unities.

1. ALGEBRAIC THEORIES

For algebraic theories à la Lawvere (1963) the reader is referred to Borceux (1994, Vol. 2, §3) and Schubert (1972, §18).

1.0. The Booleanization of Adjunctions

Let X be a Boolean locale, which shall be fixed throughout this subsection. Given two X-categories $\mathscr{C}$ and $\mathscr{D}$, their canonical X-product $\mathscr{C} \times_X \mathscr{D}$ is defined as follows:

(1.0.1) $\text{Ob}\,\mathscr{C} \times_X \mathscr{D} = \text{Ob}\,\mathscr{C} \times_X \text{Ob}\,\mathscr{D}$

(1.0.2) $\mathrm{Mor}\,\mathscr{C} \times_X \mathscr{D} = \mathrm{Mor}\,\mathscr{C} \times_X \mathrm{Mor}\,\mathscr{D}$
(1.0.3) $d_{\mathscr{C}\times_X\mathscr{D}}((f, g)) = d_{\mathscr{C}}(f) = d_{\mathscr{D}}(g)$ for all $(f, g) \in \mathrm{Mor}\,\mathscr{C} \times_X \mathrm{Mor}\,\mathscr{D}$
(1.0.4) $r_{\mathscr{C}\times_X\mathscr{D}}((f, g)) = r_{\mathscr{C}}(f) = r_{\mathscr{D}}(g)$ for all $(f, g) \in \mathrm{Mor}\,\mathscr{C} \times_X \mathrm{Mor}\,\mathscr{D}$
(1.0.5) $\mathrm{id}_{\mathscr{C}\times_X\mathscr{D}}((x, y)) = (\mathrm{id}_{\mathscr{C}}(x), \mathrm{id}_{\mathscr{D}}(y))$ for all $(x, y) \in \mathrm{Ob}\,\mathscr{C} \times_X \mathrm{Ob}\,\mathscr{D}$
(1.0.6) $(f', g') \circ_{\mathscr{C}\times_X\mathscr{D}} (f, g) = (f' \circ_{\mathscr{C}} f, g' \circ_{\mathscr{D}} g)$ for all $((f', g'), (f, g)) \in \mathrm{Mor}(\mathscr{C} \times_X \mathscr{D}) \times_{\mathrm{Ob}\,\mathscr{C}\times_X\mathscr{D}} \mathrm{Mor}(\mathscr{C} \times_X \mathscr{D})$

Let $\mathscr{C}$ and $\mathscr{D}$ be small_1 X-categories, $\mathscr{F}$ an X-functor from $\mathscr{C}$ to $\mathscr{D}$, and $\mathscr{G}$ an X-functor from $\mathscr{D}$ to $\mathscr{C}$. These entities shall be fixed throughout the remainder of this subsection.

Example 1.0.1. The assignment $(x, y) \in \mathrm{Ob}\,\mathscr{C}^{op} \times_X \mathscr{C} \mapsto (Ex, \mathscr{C}_X(x, y))$, where $\mathscr{C}_X(x, y)$ is to be regarded as an X_{Ex}-set, naturally induces an X-functor from $\mathscr{C}^{op} \times_X \mathscr{C}$ to $\mathscr{BE}ns_1(X)$, to be denoted by $\mathscr{C}(?, ??)_X$. Similarly, we have an X-functor from $\mathscr{D}^{op} \times_X \mathscr{D}$ to $\mathscr{BE}ns_1(X)$, to be denoted by $\mathscr{D}(?, ??)_X$.

Example 1.0.2. The assignment $(x, y) \in \mathrm{Ob}\,\mathscr{C}^{op} \times_X \mathscr{D} \mapsto (\mathrm{E}x, \mathscr{D}_X(\mathscr{F}x, y))$, where $\mathscr{D}_X(\mathscr{F}x, y)$ is to be put down as an X_{Ex}-set, naturally induces an X-functor from $\mathscr{C}^{op} \times_X \mathscr{D}$ to $\mathscr{BE}ns_1(X)$, to be denoted by $\mathscr{D}(\mathscr{F}?, ??)_X$. Similarly, the assignment $(x, y) \in \mathrm{Ob}\,\mathscr{C}^{op} \times_X \mathscr{D} \mapsto (\mathrm{E}x, \mathscr{C}_X(x, \mathscr{G}y))$, where $\mathscr{C}_X(x, \mathscr{G}y)$ is to be put down as an X_{Ex}-set, naturally induces an X-functor from $\mathscr{C}^{op} \times_X \mathscr{D}$ to $\mathscr{BE}ns_1(X)$, to be denoted by $\mathscr{C}(?, \mathscr{G}??)_X$.

If there exists an X-isomorphism φ between the two X-functors $\mathscr{D}(\mathscr{F}?, ??)_X$ and $\mathscr{C}(?, \mathscr{G}??)_X$ from $\mathscr{C}^{op} \times_X \mathscr{D}$ to $\mathscr{BE}ns_1(X)$, then the triple $(\mathscr{F}, \mathscr{G}, \varphi)$ is called an X-*adjunction from* $\mathscr{C}$ *to* $\mathscr{D}$, in which $\mathscr{F}$ is called an X-*left-adjoint* of $\mathscr{G}$ and $\mathscr{G}$ is called an X-*right-adjoint* of $\mathscr{F}$. The X-left-adjoint of $\mathscr{G}$ is determined uniquely by $\mathscr{G}$ up to natural X-isomorphisms so long as it exists, and the X-right-adjoint of $\mathscr{F}$ is determined uniquely by $\mathscr{F}$ up to natural X-isomorphisms so long as it exists, as can be seen simply by Booleanizing the corresponding statements of standard category theory.

1.1. The Booleanization of Algebraic Theories

Let X be a Boolean locale with $\mathscr{P}(X) = \mathbf{B}$, which shall be fixed throughout this subsection. An *algebraic* X-*theory* is a small_0 X-*category* $\mathscr{T}$ with a sequence $\{x_n\}_{n\in\mathbf{N}}$ of total objects such that:

(1.1.1) $\{x_n\}_{n\in\mathbf{N}}$ is an X-basis of the X-set $\mathrm{Ob}\,\mathscr{T}$
(1.1.2) For each $n \in \mathbf{N}$, x_n is an nth X-power of x_1

Such a sequence $\{x_n\}_{n\in\mathbf{N}}$ is unique if it exists, and it is called the *core* of the algebraic X-theory $\mathscr{T}$. Given algebraic X-theories $\mathscr{T}$ and $\mathscr{S}$ with their cores $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$, respectively, an X-*theory-morphism* from $\mathscr{T}$ to

$\mathcal{S}$ is an X-functor $\mathcal{F}$: $\mathcal{T} \to \mathcal{S}$ which preserves X-finite X-products and $\mathcal{F}(x_n) = y_n$ for all $n \in \mathbf{N}$. We denote by **BATh**(X) the category of algebraic X-theories and X-theory-morphisms. It is easy to see that if $\mathcal{A}$ is an algebraic X-theory and $p \in \mathbf{B}$, then $\mathcal{A}_p$ is an algebraic X_p-theory.

Example 1.1.1. The totality of **BATh**(X_p)'s for all $p \in \mathbf{B}$ naturally forms an X-category, to be called somewhat misleadingly the X-category of algebraic X-theories and to be denoted by $\mathcal{BATh}$(X).

Let $\mathcal{T}$ be an algebraic X-theory. We denote by **BACat**(X; $\mathcal{T}$) the category of X-functors from $\mathcal{T}$ to $\mathcal{BEns}_0$(X) preserving X-finite X-products and their natural X-transformations. The objects of **BACat**(X; $\mathcal{T}$) are called $\mathcal{T}$*-X-algebras*, and the morphisms of **BACat**(X; $\mathcal{T}$) are called $\mathcal{T}$*-homomorphisms*. The category **BACat**(X; $\mathcal{T}$) is called the *algebraic category* corresponding to $\mathcal{T}$. The notion of being finitely generated (cf. Borceux, 1994, Vol. 2, §3.8) can be easily Booleanized to yield the notion of being X-*finitely generated.* The full subcategory of X-finitely generated $\mathcal{T}$-X-algebras of **BACat**(X; $\mathcal{T}$) is denoted by $\mathbf{BACat}_{\mathrm{fg}}$(X; $\mathcal{T}$).

Example 1.1.2. The totality of **BACat**(X_p; $\mathcal{T} \lceil p$)'s for all $p \in \mathbf{B}$ naturally forms an X-category, to be called the algebraic *X*-category corresponding to $\mathcal{T}$ and to be denoted by $\mathcal{BACat}$(X; $\mathcal{T}$). The X-category $\mathcal{BACat}_{\mathrm{fg}}$(X; $\mathcal{T}$) is defined similarly.

Let $\mathcal{F}$: $\mathcal{T} \to \mathcal{S}$ be an X-theory-morphism of algebraic X-theories. Then the assignment

$$(p, \mathcal{H}) \in \mathrm{Ob}\ \mathcal{BACat}(\mathrm{X}; \mathcal{S}) \mapsto (p, \mathcal{H} \circ (\mathcal{F} \lceil p)) \in \mathrm{Ob}\ \mathcal{BACat}(\mathrm{X}; \mathcal{T})$$

naturally induces an X-functor from $\mathcal{BACat}$(X; $\mathcal{S}$) to $\mathcal{BACat}$(X; $\mathcal{T}$), to be denoted by $\mathcal{BACat}$(X; $\mathcal{F}$), for which we have the following result:

Theorem 1.1.3. The X-functor $\mathcal{BACat}$(X; $\mathcal{F}$): $\mathcal{BACat}$(X; $\mathcal{S}$) $\to$ $\mathcal{BACat}$(X; $\mathcal{T}$) has an X-left-adjoint.

Proof. Booleanize Theorem 18.5.3 of Schubert (1972). ∎

1.2. Relations Between Two Booleanized Algebraic Theories

Let f: $X_- \to X_+$ be an arbitrary morphism in **BLoc**, which shall be fixed throughout this section.

Example 1.2.1. The f-functor $\mathfrak{f}^*_{\mathcal{BEns}_0}$: $\mathcal{BEns}_0(X_+) \to \mathcal{BEns}_0(X_-)$ naturally induces an f-functor from $\mathcal{BATh}(X_+)$ to $\mathcal{BATh}(X_-)$, to be denoted by $\mathfrak{f}^*_{\mathcal{BATh}}$.

Let $\mathscr{T}_\pm$ be an $X_\pm$-algebraic theory and $\mathscr{F}$ an X-theory-morphism from $\mathfrak{f}^*_{\mathscr{BAT}}\mathscr{T}_+$ to $\mathscr{T}_-$. These entities shall be fixed throughout the remainder of this subsection.

Example 1.2.2. The $\mathfrak{f}$-functor $\mathfrak{f}^*_{\mathscr{BE}ns_0}$: $\mathscr{BE}ns_0(X_+) \to \mathscr{BE}ns_0(X_-)$ naturally induces an $\mathfrak{f}$-functor

$$\mathfrak{f}^{-1}_{\mathscr{BACat}}\colon \mathscr{BACat}(X_+; \mathscr{T}_+) \to \mathscr{BACat}(X_-; \mathfrak{f}^*_{\mathscr{BAT}}\mathscr{T}_+)$$

By Theorem 1.1.3 the X-algebraic functor

$$\mathscr{F}^b\colon \mathscr{BACat}(X_-; \mathscr{T}_-) \to \mathscr{BACat}(X_-; \mathfrak{f}^*_{\mathscr{BAT}}\mathscr{T}_+)$$

has an X-left adjoint

$$\mathscr{F}_*\colon \mathscr{BACat}(X_-; \mathfrak{f}^*_{\mathscr{BAT}}\mathscr{T}_+) \to \mathscr{BACat}(X_-; \mathscr{T}_-)$$

We denote the composite $\mathscr{F}_* \circ \mathfrak{f}^{-1}_{\mathscr{BACat}}$ by $\mathfrak{f}^*_{\mathscr{BACat}}[\mathscr{T}_-, \mathscr{T}_+, \mathscr{F}]$ or simply by $\mathfrak{f}^*_{\mathscr{BACat}}$.

As in Proposition 2.9 of our previous paper (Nishimura, 1995c), we have the following result:

Theorem 1.2.3. The $\mathfrak{f}$-functor

$$\mathfrak{f}^*_{\mathscr{BACat}}[\mathscr{T}_-, \mathscr{T}_+, \mathscr{F}]\colon \mathscr{BACat}(X_+; \mathscr{T}_+) \to \mathscr{BACat}(X_-; \mathscr{T}_-)$$

maps X_+-finite X_+-colimits to X_--colimits.

2. GROTHENDIECK TOPOSES

The first two subsections of this section give a review. The reader is referred to Makkai and Reyes (1977).

2.1. Topologies

The notion of a topology on a topos was introduced by Lawvere and Tierney as a generalization of that of Grothendieck topology approximately a quarter of a century ago. Let us recall that a *topology* on a topos **E** with a subobject classifier t: $1 \to \Omega$ is a morphism j: $\Omega \to \Omega$ abiding by the following identities:

(2.1.1) $j \circ t = t$
(2.1.2) $j \circ j = j$
(2.1.3) $j \circ \wedge = \wedge \circ (j \times j)$

A *universal closure operation* on **E** is an assignment to each subobject $x \to a$ of another subobject $\bar{x} \to a$ (called the *closure* of x in a) abiding by the following conditions:

(2.1.4) $x \subseteq \bar{x}$
(2.1.5) If $x \subseteq y$, then $\bar{x} \subseteq \bar{y}$
(2.1.6) $\bar{\bar{x}} = \bar{x}$
(2.1.7) $f^{-1}(\bar{x}) = \overline{f^{-1}(x)}$ for any morphism $f\colon b \to a$

It is well known that there is a bijection between the topologies on **E** and the universal closure operations on **E**, for which the reader is referred to Borceux (1994, Vol. 3, Proposition 9.1.3).

Example 2.1.1. Each topos **E** with a subobject classifier $1 \to \Omega$ has the weakest topology $j_{\mathbf{E}}$, namely, the identity morphism of Ω, which is called the *trivial topology* on **E**. Its corresponding universal closure operation assigns to each subobject $x \to a$ of itself.

A topos endowed with a topology is called a *localized topos*. Let $(\mathbf{E}, j)$ be a localized topos. A subobject $x \to a$ is said to be *dense* if $\bar{x} = a$. An object b of **E** is called a *j-sheaf* if for any dense subobject $s\colon x \to a$ and any morphism $f\colon x \to b$ there exists a unique morphism $g\colon a \to b$ such that $f = g \circ s$. The full subcategory of **E** whose objects are all j-sheaves is denoted by $\mathbf{Sh}(\mathbf{E}, j)$, for which the following associated sheaf functor theorem is fundamental.

Theorem 2.1.2. The inclusion functor $\mathbf{i}_j\colon \mathbf{Sh}(\mathbf{E}, j) \to \mathbf{E}$ has a left adjoint $\mathbf{a}_j\colon \mathbf{E} \to \mathbf{Sh}(\mathbf{E}, j)$, which is left exact. The category $\mathbf{Sh}(\mathbf{E}, j)$ is a topos. Therefore the pair $(\mathbf{i}_j, \mathbf{a}_j)$ forms a geometric morphism from $\mathbf{Sh}(E, j)$ to E.

For a proof of the above theorem, the reader is referred to Borceux (1994, Vol. 3, Theorems 9.2.10, 9.2.11, and 9.3.8).

A geometric morphism $\varphi = (\varphi_*, \varphi^*)\colon \mathbf{E}_- \to \mathbf{E}_+$ is said to be *prelocalized* if both of the toposes $\mathbf{E}_\pm$ are localized (with topologies $j_\pm$). A prelocalized geometric morphism $\varphi = (\varphi_*, \varphi^*)\colon (\mathbf{E}_-, j_-) \to (\mathbf{E}_+, j_+)$ is said to be *localized* if φ^* satisfies the following condition:

(2.1.8) $\varphi^*\bar{x} \subseteq \overline{\varphi^* x}$ for any subobject $x \to a$ in $\mathbf{E}_+$.

Example 2.1.3. For each localized topos $(\mathbf{E}, j)$, the pair $(\mathbf{I}_\mathbf{E}, \mathbf{I}_\mathbf{E})$ with $\mathbf{I}_\mathbf{E}$ the identity functor of **E** is a localized geometric morphism from $(\mathbf{E}, j)$ to $(\mathbf{E}, j_\mathbf{E})$, where $j_\mathbf{E}$ is the trivial topology on **E** discussed in Example 2.1.1.

Theorem 2.1.4. Let $\varphi = (\varphi_*, \varphi^*)\colon (\mathbf{E}_-, j_-) \to (\mathbf{E}_+, j_+)$ be a localized geometric morphism. Then φ_* preserves sheaves. That is, $x \in \mathrm{Ob}\, \mathbf{Sh}(\mathbf{E}_-,$

j_-) always implies $\varphi_* x \in \mathrm{Ob}\ \mathbf{Sh}(\mathbf{E}_+, j_+)$, so that φ_* induces a functor $\tilde{\varphi}_*$: $\mathbf{Sh}(\mathbf{E}_-, j_-) \to \mathbf{Sh}(\mathbf{E}_+, j_+)$ by restriction.

Proof. If $s: x \to a$ is a dense monic in $\mathbf{E}_+$, then $\varphi^*: \varphi^*\mathbf{x} \to \varphi^* a$ is also a dense monic in $\mathbf{E}_-$, for φ^* is left exact and φ is localized. Since $\varphi^* \dashv \varphi_*$, the following square is commutative for each $b \in \mathrm{Ob}\ \mathbf{E}_-$:

$$\begin{array}{ccc} \mathbf{E}_+(a, \varphi_* b) & \cong & \mathbf{E}_-(\varphi^* a, b) \\ {\scriptstyle \mathbf{E}_+(s,\, \varphi_* b)} \downarrow & & \downarrow {\scriptstyle \mathbf{E}_-(\varphi^* s,\, b)} \\ \mathbf{E}_+(x, \varphi_* b) & \cong & \mathbf{E}_-(\varphi^* x, b) \end{array}$$

Thus, if b is a j_--sheaf so that $\mathbf{E}_-(\varphi^* s, b)$ is bijective, then $\mathbf{E}_+(s, \varphi_* b)$ is also bijective. Therefore we are sure that $\varphi_* b \in \mathrm{Ob}\ \mathbf{Sh}(\mathbf{E}_+, j_+)$ whenever $b \in \mathrm{Ob}\ \mathbf{Sh}(\mathbf{E}_-, j_-)$. ■

Theorem 2.1.5. Let $\varphi = (\varphi_*, \varphi^*): (\mathbf{E}_-, j_-) \to (\mathbf{E}_+, j_+)$ be a localized geometric morphism. Then the functor

$$\tilde{\varphi}^* = \mathbf{a}_{j_-} \circ \varphi^* \circ \mathbf{i}_{j_+}: \mathbf{Sh}(\mathbf{E}_+, j_+) \to \mathbf{Sh}(\mathbf{E}_-, j_-)$$

is left exact and is left adjoint to the functor $\tilde{\varphi}_*$: $\mathbf{Sh}(\mathbf{E}_-, j_-) \to \mathbf{Sh}(\mathbf{E}_+, j_+)$. Therefore the pair $\tilde{\varphi} = (\tilde{\varphi}_*, \tilde{\varphi}^*)$ constitutes a geometric morphism from $\mathbf{Sh}(\mathbf{E}_-, j_-)$ to $\mathbf{Sh}(\mathbf{E}_+, j_+)$.

Proof. The functor $\tilde{\varphi}^*$ is surely left exact, since all of $\mathbf{a}_{j_-}$, φ^*, and $\mathbf{i}_{j_+}$ are left exact. Let $x \in \mathrm{Ob}\ \mathbf{Sh}(\mathbf{E}_+, j_+)$ and $y \in \mathrm{Ob}\ \mathbf{Sh}(\mathbf{E}_-, j_-)$. Then we have

$$\begin{aligned} \mathbf{Sh}(\mathbf{E}_-, j_-)(\tilde{\varphi}^* x, y) &= \mathbf{Sh}(\mathbf{E}_-, j_-)((\mathbf{a}_{j_-} \circ \varphi^* \circ \mathbf{i}_{j_+})x, y) \\ &\cong \mathbf{E}_-((\varphi^* \circ \mathbf{i}_{j_+})x, \mathbf{i}_{j_-} y) \\ &\cong \mathbf{E}_+(\mathbf{i}_{j_+} x, (\varphi_* \circ \mathbf{i}_{j_-})y) \\ &\cong \mathbf{Sh}(\mathbf{E}_+, j_+)(x, (\mathbf{a}_{j_+} \circ \varphi_* \circ \mathbf{i}_{j_-})y) \\ &= \mathbf{Sh}(\mathbf{E}_+, j_+)(x, \tilde{\varphi}_* y) \end{aligned}$$

Therefore $\tilde{\varphi}^* \dashv \tilde{\varphi}_*$. ■

Theorem 2.1.6. Let $\varphi = (\varphi_*, \varphi^*): (\mathbf{E}_1, j_1) \to (\mathbf{E}_2, j_2)$ and $\psi = (\psi_*, \psi^*)$: $(\mathbf{E}_2, j_2) \to (\mathbf{E}_3, j_3)$ be localized geometric morphisms. Let $\chi = \psi \circ \varphi$, so that $\chi_* = \psi_* \circ \varphi_*$ and $\chi^* = \varphi^* \circ \psi^*$. Then $\tilde{\chi} = \tilde{\psi} \circ \tilde{\varphi}$ up to isomorphic conjugates. In particular, $\tilde{\chi}^* = \tilde{\varphi}^* \circ \tilde{\psi}^*$ up to natural isomorphisms.

Proof. By the very definitions of $\tilde{\varphi}_*$, $\tilde{\psi}_*$, and $\tilde{\chi}_*$ under Theorem 2.1.4, we have $\tilde{\chi}_* = \tilde{\psi}_* \circ \tilde{\varphi}_*$. Since the left-adjoint of a functor, if it exists, is determined uniquely up to natural isomorphisms (MacLane, 1971, Chapter IV. §1, Corollary 1 of Theorem 2), the desired conclusion follows readily. ■

As a corollary of this theorem, we have the following result.

Theorem 2.1.7. Let $\theta = (\theta_*, \theta^*)$: $(\mathbf{E}_-, j_-) \to (\mathbf{E}_+, j_+)$ be a localized geometric morphism. Then the functors $\mathbf{a}_{j_-} \circ \theta^*$ and $\mathbf{a}_{j_-} \circ \theta^* \circ \mathbf{i}_{j_+} \circ \mathbf{a}_{j_+}$ from $(\mathbf{E}_+, j_+)$ to $\mathbf{Sh}(\mathbf{E}_-, j_-)$ are naturally isomorphic.

Proof. By taking (θ_*, θ^*): $(\mathbf{E}_-, j_-) \to (\mathbf{E}_+, j_+)$ and $(\mathbf{I}_{\mathbf{E}_+}, \mathbf{I}_{\mathbf{E}_+})$: $(\mathbf{E}_+, j_+) \to (\mathbf{E}_+, j_{\mathbf{E}_+})$ for (φ_*, φ^*): $(\mathbf{E}_1, j_1) \to (\mathbf{E}_2, j_2)$ and (ψ_*, ψ^*): $(\mathbf{E}_2, j_2) \to (\mathbf{E}_3, j_3)$, respectively, in the above theorem, we get the desired result. ■

2.2. Grothendieck Topologies

Recall that a *Grothendieck topology* on a category $\mathbf{C}$ is an assignment $\mathbf{L}$ to each $a \in \mathrm{Ob}\,\mathbf{C}$ of a family $\mathbf{L}(a)$ of subfunctors of $\mathbf{C}(?, a)$ yielding the following conditions:

(2.2.1) $\mathbf{C}(?, a) \in \mathbf{L}(a)$.

(2.2.2) Let f: $b \to a$ be a morphism of $\mathbf{C}$. Let $\mathbf{R}$ and $\mathbf{R}_f$ be subfunctors of $\mathbf{C}(?, a)$ and $\mathbf{C}(?, b)$ respectively. If the square

$$\begin{array}{ccc} \mathbf{R}_f & \longrightarrow & \mathbf{R} \\ \downarrow & & \downarrow \\ \mathbf{C}(?, b) & \xrightarrow[\mathbf{C}(?, f)]{} & \mathbf{C}(?, a) \end{array}$$

is a pullback diagram and $\mathbf{R} \in \mathbf{L}(a)$, then $\mathbf{R}_f \in \mathbf{L}(b)$.

(2.2.3) Let $a \in \mathrm{Ob}\,\mathbf{C}$, $\mathbf{R}$ a subfunctor of $\mathbf{C}(?, a)$, and $\mathbf{S} \in \mathbf{L}(a)$. If for any $b \in \mathrm{Ob}\,\mathbf{C}$ and any f: $b \to a \in Sb$ we have $\mathbf{R}_f \in \mathbf{L}(b)$ with $\mathbf{R}_f$ being defined as in (2.2.2), then we have $\mathbf{R} \in \mathbf{L}(a)$.

A subfunctor $\mathbf{R}$ of $\mathbf{C}(?, a)$ for some $a \in \mathrm{Ob}\,\mathbf{C}$ is usually identified with a *sieve* on a, which is by definition a set of morphisms f of $\mathbf{C}$ with codomain a which is closed under the right composition. Given a Grothendieck topology $\mathbf{L}$ on $\mathbf{C}$ and $a \in \mathrm{Ob}\,\mathbf{C}$, a set S of morphisms of $\mathbf{C}$ with codomain a is said to $\mathbf{L}$-*cover* a if $\mathrm{Siv}(S) = \{g\colon b \to a \in \mathrm{Mor}\,\mathbf{C} \mid g = f \circ h$ for some $f \in S$ and some $h \in \mathrm{Mor}\,\mathbf{C}\} \in \mathbf{L}(a)$. The totality of sets of morphisms of $\mathbf{C}$ with codomain a which $\mathbf{L}$-cover a is denoted by $\mathrm{Cov}_{\mathbf{L}}(a)$.

The following well-known theorem signifies that the notion of a topology on a topos is a good generalization of a Grothendieck topology on a category.

Theorem 2.2.1. Let **C** be a small$_1$ category. Then there is a bijection between the topologies on the topos **PreSh(C)** and the Grothendieck topologies on the category **C**.

For a proof of the above theorem the reader is referred, e.g., to Proposition 9.1.2 of Borceux (1994, Vol. 3). A pair (**C**, **L**) of a small$_1$ category **C** and a Grothendieck topology **L** on **C** is called a *site*. For a site (**C**, **L**), the topology on **PreSh(C)** corresponding to the Grothendieck topology **L** under Theorem 2.2.1 is denoted by j[**L**] and the topos **Sh**(**PreSh(C)**, j[**L**]) is denoted by **Sh(C, L)**. For **X** $\in$ **PreSh(C)** and a subfunctor **Y** of **X**, the closure $\overline{\mathbf{Y}}$ of **Y** in **X** with respect to the topology j[**L**] can be calculated as follows:

(2.2.4) For each $a \in \text{Ob}\ \mathbf{C}$, $\overline{\mathbf{Y}}a$ consists of all $x \in \mathbf{X}a$ such that $\epsilon(x, \mathbf{Y}) = \{f: b \to a \in \text{Mor}\ \mathbf{C} \mid \mathbf{X}f(x) \in \mathbf{Y}b\} \in \mathbf{L}(a)$

For a site (**C**, **L**) it is easy to see the following result.

Lemma 2.2.2. Let $a \in \text{Ob}\ \mathbf{C}$ and **S** be a subfunctor of **C**(?, a). Then $\mathbf{S} \in \mathbf{L}(a)$ iff $\text{id}(a) \in \overline{\mathbf{S}}a$.

Proof. We note that

$$\mathbf{S} = \{f: b \to a \in \text{Mor}\ \mathbf{C} \mid \mathbf{C}(f, A)(\text{id}(a)) \in \mathbf{S}b\} = \epsilon(\text{id}(a), \mathbf{S})$$

since $\mathbf{C}(f, a)((\text{id}(a)) = f$, which establishes the desired result forthwith. ■

Corollary 2.2.3. Let a and **S** be the same as in Lemma 2.2.2. Then $\mathbf{S} \in \mathbf{L}(a)$ iff $\overline{\mathbf{S}} = \mathbf{C}(?, a)$.

Proof. This follows from the above lemma and the simple fact that for any subfunctor **T** of **C**(?, a), $\mathbf{T} = \mathbf{C}(?, a)$ iff $\text{id}(a) \in \mathbf{T}$. ■

It is also easy to see the following.

Lemma 2.2.4. Let **F**: **D** → **PreSh(C)** be a functor, **X** $\in$ Ob **PreSh(C)**, **Y** be a subfunctor of **X**, and σ: **F** → $\langle \mathbf{X} \rangle_{\mathbf{D}}$ be a colimit. Let $\mathbf{F_Y}$ be the functor from **D** to **PreSh(C)** such that $\mathbf{F_Y}(d) = \sigma(d)^{-1}(\mathbf{Y})$ for any $d \in \text{Ob}\ \mathbf{D}$ and the diagram

$$\begin{array}{ccc} \mathbf{F_Y}d_1 & \longrightarrow & \mathbf{F}d_1 \\ {\scriptstyle \mathbf{F_Y}f}\downarrow & & \downarrow{\scriptstyle \mathbf{F}f} \\ \mathbf{F_Y}d_2 & \longrightarrow & \mathbf{F}d_2 \end{array}$$

is commutative for any $f\colon d_1 \to d_2 \in \mathrm{Mor}\,\mathbf{D}$. Then the natural transformation $\sigma_{\mathbf{Y}}\colon \mathbf{F}_{\mathbf{Y}} \to \langle \mathbf{Y}\rangle_{\mathbf{D}}$ making the square

$$\begin{array}{ccc} \mathbf{F}_{\mathbf{Y}}d & \xrightarrow{\sigma_{\mathbf{Y}}(d)} & \mathbf{Y} \\ \downarrow & & \downarrow \\ \mathbf{F}d & \xrightarrow[\sigma(d)]{} & \mathbf{X} \end{array}$$

a pullback diagram for any $d \in \mathrm{Ob}\,\mathbf{D}$ is a colimit. Simply by replacing $\mathbf{Y}$ by $\overline{\mathbf{Y}}$ in the above definitions of $\mathbf{F}_{\mathbf{Y}}$ and $\sigma_{\mathbf{Y}}$, we get a functor $\mathbf{F}_{\overline{\mathbf{Y}}}\colon \mathbf{D} \to \mathbf{PreSh}(\mathbf{C})$ and a colimit $\sigma_{\overline{\mathbf{Y}}}\colon \mathbf{F}_{\overline{\mathbf{Y}}} \to \langle \overline{\mathbf{Y}}\rangle_{\mathbf{D}}$. Then $\mathbf{F}_{\mathbf{Y}}(d)$ is dense in $\mathbf{F}_{\overline{\mathbf{Y}}}(d)$ and $\mathbf{F}_{\overline{\mathbf{Y}}}(d)$ is closed in $\mathbf{F}(d)$ for any $d \in \mathrm{Ob}\,\mathbf{D}$. ■

Theorem 2.2.5. Let $\mathbf{F}\colon \mathbf{C}_+ \to \mathbf{C}_-$ be a left-exact functor of small$_1$ categories with $\mathbf{C}_+$ being finitely complete. Then the functor $\varphi_*[\mathbf{F}]\colon \mathbf{PreSh}(\mathbf{C}_-) \to \mathbf{PreSh}(\mathbf{C}_+)$ assigning to each $\mathbf{X} \in \mathrm{Ob}\,\mathbf{PreSh}(\mathbf{C}_-)$ of $\mathbf{X} \circ \mathbf{F} \in \mathrm{Ob}\,\mathbf{PreSh}(\mathbf{C}_+)$ and assigning to each $\tau \in \mathrm{Mor}\,\mathbf{PreSh}(\mathbf{C}_-)$ of $\tau \circ \mathbf{F} \in \mathrm{Mor}\,\mathbf{PreSh}(\mathbf{C}_+)$ has a left adjoint $\varphi^*[F]\colon \mathbf{PreSh}(\mathbf{C}_+) \to \mathbf{PreSh}(\mathbf{C}_-)$, which is left exact.

Proof. See Theorem 17.1.6 of Schubert (1972). ■

A left-exact functor $\mathbf{F}\colon \mathbf{C}_+ \to \mathbf{C}_-$ of small$_1$ categories with $\mathbf{C}_+$ being finitely complete is called a *topological functor* if the categories $\mathbf{C}_\pm$ are endowed with Grothendieck topologies $\mathbf{L}_\pm$, respectively. In this case we preferably write $\mathbf{F}\colon (\mathbf{C}_+, \mathbf{L}_+) \to (\mathbf{C}_-, \mathbf{L}_-)$. A topological functor $\mathbf{F}\colon (\mathbf{C}_+, \mathbf{L}_+) \to (\mathbf{C}_-, \mathbf{L}_-)$ is said to be *continuous* if $\mathbf{S} \in \mathrm{Cov}_{\mathbf{L}_+}(a)$.

Let $\mathbf{F}\colon (\mathbf{C}_+, \mathbf{L}_+) \to (\mathbf{C}_-, \mathbf{L}_-)$ be a topological functor, which shall be fixed throughout the remainder of this subsection.

Lemma 2.2.6. Let $a \in \mathrm{Ob}\,\mathbf{C}_+$ and $\mathbf{S}$ be a subfunctor of $\mathbf{C}_+(?, a)$. Then $\varphi^*[\mathbf{F}](\mathbf{S}) = \mathrm{Siv}(\mathbf{FS})$.

Proof. This follows easily from a pointwise formula for Kan extension such as formula (10) of MacLane (1971, Chapter X, §3). ■

Theorem 2.2.7. The prelocalized geometric morphism

$$(\varphi_*[\mathbf{F}], \varphi^*[\mathbf{F}])\colon (\mathbf{PreSh}(\mathbf{C}_-), j[\mathbf{L}_-]) \to (\mathbf{PreSh}(\mathbf{C}_+), j[\mathbf{L}_+])$$

is localized iff for any $a \in \mathrm{Ob}\,\mathbf{C}_+$ and any subfunctor $\mathbf{S}$ of $\mathbf{C}(?, a)$, $\overline{\mathbf{S}} = \mathbf{C}_+(?, a)$ implies $\overline{\varphi^*[\mathbf{F}](\mathbf{S})} = \mathbf{C}_-(?, \mathbf{F}a)$.

Proof. The only-if part is obvious. To see the if part, let $\mathbf{X} \in \mathbf{PreSh}(\mathbf{C}_+)$ and Y be a subfunctor of $\mathbf{X}$. We must show that $\varphi^*[\mathbf{F}]\overline{\mathbf{Y}} \subseteq \overline{\varphi^*[\mathbf{F}]\mathbf{Y}}$, where $\overline{\mathbf{Y}}$ denotes the closure of $\mathbf{Y}$ in $\mathbf{X}$ and $\overline{\varphi^*[\mathbf{F}]\mathbf{Y}}$ denotes the closure of $\varphi^*[\mathbf{F}]\mathbf{Y}$

in $\varphi^*[\mathbf{F}]\mathbf{X}$. We can and indeed do assume without loss of generality that $\mathbf{Y}$ is dense in $\mathbf{X}$, so that we have to show that $\varphi^*[\mathbf{F}]\mathbf{Y}$ is dense in $\varphi^*[\mathbf{F}]\mathbf{X}$. Since the Yoneda embedding $\mathbf{y}: \mathbf{C}_+ \to \mathbf{PreSh}(\mathbf{C}_+)$ is a dense functor by Corollary 3 of MacLane (1971, Chapter X, §6) and the functor $\varphi^*[\mathbf{F}]$ preserves arbitrary colimits and finite limits, Lemma 2.2.4 guarantees the desired general statement to be reducible to the special case that $\mathbf{X}$ is of the form $\mathbf{C}_+(?, a)$ for some $a \in \mathrm{Ob}\, \mathbf{C}_+$. This completes the proof. ∎

Theorem 2.2.8. The prelocalized geometric morphism

$$(\varphi_*[\mathbf{F}], \varphi^*[\mathbf{F}]): (\mathbf{PreSh}(\mathbf{C}_-), j[\mathbf{L}_-]) \to (\mathbf{PreSh}(\mathbf{C}_+), j[\mathbf{L}_+])$$

is localized iff the topological functor $\mathbf{F}: (\mathbf{C}_+, \mathbf{L}_+) \to (\mathbf{C}_-, \mathbf{L}_-)$ is continuous.

Proof. This follows readily from Corollary 2.2.3, Lemma 2.2.6, and Theorem 2.2.7. ∎

2.3. The Booleanization of Topologies

Let X be a Boolean locale with $\mathbf{B} = \mathscr{P}(\mathrm{X})$, which shall be fixed throughout this subsection. Recall that an X-category $\mathscr{E}$ is called an X-*topos* if its 1-slice $\mathscr{E}[1]$ is a topos, in which it is easy to see that $\mathscr{E}[p]$ is a topos for each $p \in \mathbf{B}$ and that if we denote by Ω a subobject classifier of $\mathscr{E}[1]$, then $\Omega \lceil p$ is a subobject classifier of $\mathscr{E}[p]$ for each $p \in \mathbf{B}$.

An X-topos $\mathscr{E}$ shall be fixed throughout the remainder of this subsection. An X-*topology* on $\mathscr{E}$ is a topology $\mathscr{j}$ on the topos $\mathscr{E}[1]$, which naturally induces a topology $\mathscr{j} \lceil p$ on $\mathscr{E}[p]$ for each $p \in \mathbf{B}$. The ordered pair $(\mathscr{E}, \mathscr{j})$ is called a *localized* X-*topos*. An X-*universal closure operation* on $\mathscr{E}$ is a universal closure operation on $\mathscr{E}[1]$, which naturally induces a universal closure operation on $\mathscr{E}[p]$ for each $p \in \mathbf{B}$ such that for each subobject s: $x \to a$ of $\mathscr{E}[1]$ and each $p \in \mathbf{B}$, $\bar{x} \lceil p = \overline{x \lceil p}$. The well-known bijection between the topologies and the universal closure operations on a topos naturally implies a bijection on the X-topologies and the X-universal closure operations on the X-topos $\mathscr{E}$.

Let $\mathscr{j}$ be a topology on $\mathscr{E}$. The totality of $\mathbf{Sh}(\mathscr{E}[p], \mathscr{j}[p])$ for all $p \in \mathbf{B}$ is easily seen to form an X-subcategory of $\mathscr{E}$, which is an X-topos and is denoted by $\mathscr{Sh}(\mathscr{E}, \mathscr{j})$. By simply Booleanizing Theorem 2.1.2, we have the following result.

Theorem 2.3.1. Let $(\mathscr{E}, \mathscr{j})$ be a localized X-topos. Then the inclusion X-functor $i_{\mathscr{j}}: \mathscr{Sh}(\mathscr{E}, \mathscr{j}) \to \mathscr{E}$ has an X-left-adjoint $a_{\mathscr{j}}: \mathscr{E} \to \mathscr{Sh}(\mathscr{E}, \mathscr{j})$, which is left X-exact.

2.4. The Booleanization of Grothendieck Topologies

Let X be a Boolean locale with $\mathscr{P}(\mathrm{X}) = \mathbf{B}$, which shall be fixed throughout this subsection. A *Grothendieck* X-*topology* on a small_1 X-category $\mathscr{C}$ is an X-function $\mathscr{L}$ from Ob $\mathscr{C}$ to Ob $\mathscr{BEns}_1(\mathrm{X})$ abiding by the following conditions:

(2.4.1) For each $a \in \mathrm{Ob}\,\mathscr{C}$, every element of $\mathscr{L}(a)$ is a partial $\mathrm{X}_{\mathrm{E}a}$-subfunctor of $\mathscr{C}^{\mathrm{X}}(?, a)$.

(2.4.2) $\mathscr{C}^{\mathrm{X}}(?, a) \in \mathscr{L}(a)$.

(2.4.3) Let $f\colon b \to a \in \mathrm{Mor}\,\mathscr{C}$. Let $\mathscr{R}$ and $\mathscr{R}_f$ be $\mathrm{X}_{\mathrm{E}a}$-subfunctors of $\mathscr{C}^{\mathrm{X}}(?, a)$ and $\mathscr{C}^{\mathrm{X}}(?, b)$, respectively. If the square

$$\begin{array}{ccc} \mathscr{R}_f & \longrightarrow & \mathscr{R} \\ \downarrow & & \downarrow \\ \mathscr{C}^{\mathrm{X}}(?, b) & \xrightarrow[\mathscr{C}^{\mathrm{X}}(?, f)]{} & \mathscr{C}^{\mathrm{X}}(?, a) \end{array}$$

is a pullback diagram of $\mathscr{BEns}_1(\mathrm{X})[\mathrm{E}a]$ and $\mathscr{R} \in \mathscr{L}(a)$, then $\mathscr{R}_f \in \mathscr{L}(b)$.

(2.4.4) Let $a \in \mathrm{Ob}\,\mathscr{C}$, $\mathscr{R}$ be an $\mathrm{X}_{\mathrm{E}a}$-subfunctor of $\mathscr{C}^{\mathrm{X}}(?, a)$, and total $\mathscr{S} \in \mathscr{L}(a)$. If for any $b \in \mathrm{Ob}\,\mathscr{C}$ and any $f\colon b \to a \in Sb$ we have $\mathscr{R}_f \in \mathscr{L}(b)$ with $\mathscr{R}_f$ being defined as in (2.4.3), then we have $\mathscr{R} \in \mathscr{L}(a)$.

An $\mathrm{X}_{\mathrm{E}a}$-subfunctor $\mathscr{R}$ of $\mathscr{C}^{\mathrm{X}}(?, a)$ for some $a \in \mathrm{Ob}\,\mathbf{C}$ is usually identified with an $\mathrm{X}_{\mathrm{E}a}$-*sieve* on a, which is by definition an $\mathrm{X}_{\mathrm{E}a}$-set $\underline{\mathscr{R}}$ consisting of morphisms of $\mathscr{C}$:

(2.4.5) For each $f \in \underline{\mathscr{R}}$, the codomain of f is $a \lceil p$ for some $p \leq \mathrm{E}a$.

(2.4.6) The underlying set of the $\mathrm{X}_{\mathrm{E}a}$-set $\underline{\mathscr{R}}$ is a sieve on a.

In the sequel the $\mathrm{X}_{\mathrm{E}a}$-subfunctor of $\mathscr{C}^{\mathrm{X}}(?, a)$ $\mathscr{R}$ and its corresponding $\mathrm{X}_{\mathrm{E}a}$-sieve $\underline{\mathscr{R}}$ on a are identified, so that $\underline{\mathscr{R}}$ is denoted simply by the same symbol $\mathscr{R}$.

By simply Booleanizing Theorem 2.2.1, we have the following result.

Theorem 2.4.1. Let $\mathscr{C}$ be a small_1 X-category. Then there is a bijection between the X-topologies on the X-topos $\mathscr{BPreSh}(\mathrm{X}; \mathscr{C})$ and the Grothendieck X-topologies on the X-category $\mathscr{C}$.

A pair $(\mathscr{C}, \mathscr{L})$ of a small_1 X-category $\mathscr{C}$ and a Grothendieck topology $\mathscr{L}$ on $\mathscr{C}$ is called an X-*site*. For an X-site $(\mathscr{C}, \mathscr{L})$, the topology on $\mathscr{PreSh}(\mathrm{X}; \mathscr{C})$ corresponding to the Grothendieck X-topology $\mathscr{L}$ under Theorem 2.4.1

is denoted by $\not{\jmath}\,[\mathscr{L}]$ and the X-topos $\mathscr{Sh}(X; \mathscr{PreSh}(\mathscr{C}), \not{\jmath}\,[\mathscr{L}])$ is denoted by $\mathscr{Sh}(X; \mathscr{C}, \mathscr{L})$.

2.5. Relations Between Two Booleanized Grothendieck Toposes

Let $f: X_- \to X_+$ be a morphism of **BLoc**, which shall be fixed throughout this subsection.

Theorem 2.5.1. Given small_1 $X_\pm$-sets $\mathscr{V}_\pm$, there is a bijection between the **f**-functions from $\mathscr{V}_+$ to $\mathscr{V}_-$ and the X_--functions from $f^*\mathscr{V}_+$ to $\mathscr{V}_-$.

Proof. It is easy to see that there is a bijection between the **f**-functions from $\mathscr{V}_+$ to $\mathscr{V}_-$ and the morphisms from $\underline{f}^*\mathscr{V}_+$ to $\mathscr{V}_-$ in the category $\underline{\mathbf{BEns}}_1(X)$. The adjunction from $\underline{\mathbf{BEns}}_1(X)$ to $\text{BEns}_1(X)$ discussed in Nishimura (1995b, Theorem 1.2) gives a bijection $\underline{\mathbf{BEns}}_1(X)(\underline{f}^*\mathscr{V}_+, \mathscr{V}_-) \cong \mathbf{BEns}_1(X)(f^*\mathscr{V}_+, \mathscr{V}_-)$. Therefore the desired conclusion follows. ∎

By the same token, we have the following result.

Theorem 2.5.2. Given small_1 $X_\pm$-categories $\mathscr{C}_\pm$, there is a bijection between the f-functions from $\mathscr{C}_+$ to $\mathscr{C}_-$ and the X_--functors from $f^*\mathscr{C}_+$ to $\mathscr{C}_-$.

The X_--functor corresponding to an f-functor $\mathscr{F}: \mathscr{C}_+ \to \mathscr{C}_-$ in the above theorem is denoted by $\mathscr{F}_{X_-}$, while the f-functor corresponding to an X_--functor $\mathscr{G}: f^*\mathscr{C}_+ \to \mathscr{C}_-$ under the above theorem is denoted by $\mathscr{G}_f$.

It is easy to see the following.

Lemma 2.5.3. For any f-functor $\mathscr{F}: \mathscr{C}_+ \to \mathscr{C}_-$, any X_+-functor $\mathscr{H}: \mathscr{D}_+ \to \mathscr{C}_+$, and any X_--functor $\mathscr{K}: \mathscr{C}_- \to \mathscr{D}_-$, we have $(\mathscr{K} \circ \mathscr{F} \circ \mathscr{H})_{X_-} = \mathscr{K} \circ \mathscr{F}_{X_-} \circ f^*\mathscr{H}$.

Example 2.5.4. Let $\mathscr{C}_+$ be a small_1 X_+-category. The assignment

$$\mathscr{K} \in \text{Ob}\, \mathscr{BPreSh}(X_+; \mathscr{C}_+) \mapsto f^*\mathscr{K} \in \text{Ob}\, \mathscr{BPreSh}(X_-; f^*\mathscr{C}_+)$$

naturally induces an f-functor, which is to be denoted by $f^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$. For any $x \in \text{Ob}\,\mathscr{C}$ such that $E\mathscr{K} = Ex$, $(f^*_{\mathscr{BPreSh}}[\mathscr{C}_+]\mathscr{K})(f^*x) = f^*(\mathscr{K}x)$.

Theorem 2.5.5. In the above example, the f-functor $f^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$ maps small_1 X_+-colimits to X_--colimits and maps small_1 X_+-limits to X_--limits.

Proof. The Booleanization of Schubert (1972, Item 8.5.1) guarantees that X_+-colimits in $\mathscr{BPreSh}(X_+; \mathscr{C}_+)$ and X_--colimits in $\mathscr{BPreSh}(X_-; f^*\mathscr{C}_+)$ can be computed componentwise. Since $f^*_{\mathscr{BEns}_1}$ maps small_1 X_+-colimits to X_--colimits, the desired first half of the theorem follows. The remaining half of the theorem can be dealt with similarly. ∎

Theorem 2.5.6. Let $\mathscr{F}$ be an f-functor from a small$_1$ X_+-category $\mathscr{C}_+$ to a small$_1$-X_--complete X_--category $\mathscr{D}_-$. Then there is, up to natural f-isomorphisms, a unique f-functor $\mathscr{G}$: $\mathscr{BPreSh}(X_+; \mathscr{C}_+) \to \mathscr{D}_-$ mapping small$_1$ X_+-colimits to X_--colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{BPreSh}(X_+; \mathscr{C}_+) & \xrightarrow{\mathscr{G}} & \mathscr{D}_- \\ {\scriptstyle y^*}\uparrow & \nearrow_{\mathscr{F}} & \\ \mathscr{C}_+ & & \end{array}$$

Proof. The uniqueness part is obvious, since every object of $\mathscr{BPreSh}(X_+; \mathscr{C}_+)$ is an X_+-colimit of the image of a small$_1$ partial X_+-diagram in $\mathscr{C}_+$ under the Yoneda embedding **y**. By Booleanizing MacLane and Moerdijk (1992, Chapter I, §5, Corollary 4 of Theorem 2), we can see that there is an X_--functor $\mathscr{K}$ preserving small$_1$ X_--colimits and making the diagram

$$\begin{array}{ccc} \mathscr{BPreSh}(X_-; f^*\mathscr{C}_+) & \xrightarrow{\mathscr{K}} & \mathscr{D}_- \\ {\scriptstyle y^*}\uparrow & \nearrow_{\mathscr{F}_{X_-}} & \\ f^*\mathscr{C}_+ & & \end{array}$$

commutative. The desired $\mathscr{G}$ can be obtained as $\mathscr{K} \circ f^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$. ∎

Theorem 2.5.7. Let $\mathscr{F}$ be an **f**-functor from a small$_1$ X_+-category $\mathscr{C}$ to a small$_1$ X_--category $\mathscr{C}_-$. Then there is, up to natural f-isomorphisms, a unique f-functor $\pi^*[\mathscr{F}]$: $\mathscr{BPreSh}(X_+; \mathscr{C}_+) \to \mathscr{BPreSh}(X_-; \mathscr{C}_-)$ mapping small$_1$ X_+-colimits to X_--colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{BPreSh}(X_+; \mathscr{C}_+) & \xrightarrow{\pi^*[\mathscr{F}]} & \mathscr{BPreSh}(X_-; \mathscr{C}_-) \\ {\scriptstyle y^*}\uparrow & & \uparrow{\scriptstyle y^*} \\ \mathscr{C}_+ & \xrightarrow[\mathscr{F}]{} & \mathscr{C}_- \end{array}$$

Proof. Take $\mathscr{BPreSh}(X_-; \mathscr{C}_-)$ for $\mathscr{D}_-$ and $y^* \circ \mathscr{F}$: $\mathscr{C}_+ \to \mathscr{BPreSh}(X_-, \mathscr{C}_-)$ for $\mathscr{F}$: $\mathscr{C}_+ \to \mathscr{D}_-$ in the above theorem. ∎

Example 2.5.8. In the case that $X_+ = X_-$ (which we denote by X) and f is the identity functor of X, $\mathscr{F}$: $\mathscr{C}_+ \to \mathscr{C}_-$ in the above theorem is an X-functor. Therefore, under the additional assumption that the X-category $\mathscr{C}_+$ is X-finitely X-complete and $\mathscr{F}$ is X-left-exact, we have

$$\pi[\mathscr{F}] = (\pi_*[\mathscr{F}], \pi^*[\mathscr{F}]): \mathscr{BPreSh}(X; \mathscr{C}_-) \to \mathscr{BPreSh}(X; \mathscr{C}_+)$$

with $\pi_*[\mathscr{F}]$ consisting of the assignments

$$\mathscr{G} \in \mathrm{Ob}\, \mathscr{BPreSh}(X; \mathscr{C}_+) \mapsto \mathscr{G} \circ (\mathscr{F} \lceil E\mathscr{G}) \in \mathrm{Ob}\, \mathscr{BPreSh}(X; \mathscr{C}_-)$$

and

$$\sigma \in \mathrm{Mor}\, \mathscr{BPreSh}(X; \mathscr{C}_+) \mapsto \sigma \circ (\mathscr{F} \lceil E\sigma) \in \mathrm{Mor}\, \mathscr{BPreSh}(X; \mathscr{C}_-)$$

is an X-geometric morphism. This is only a Booleanization of Theorem 2.2.5. ■

Theorem 2.5.9. Let $\mathbf{g}$: X $\to X_2$ and $\mathbf{h}$: $X_2 \to X_3$ be morphisms of **BLoc**. Let $\mathscr{G}: \mathscr{C}_2 \to \mathscr{C}_1$ be a small_1 $\mathbf{g}$-functor and $\mathscr{H}: \mathscr{C}_3 \to \mathscr{C}_2$ be a small_1 $\mathbf{h}$-functor. Then the $\mathbf{h} \circ \mathbf{g}$-functors $\pi^*[\mathscr{G} \circ \mathscr{H}]$ and $\pi^*[\mathscr{G}] \circ \pi^*[\mathscr{H}]$ from $\mathscr{BPreSh}$ $(X_3; \mathscr{C}_3)$ to $\mathscr{BPreSh}(X_1; \mathscr{C}_1)$ are naturally $\mathbf{h} \circ \mathbf{g}$-isomorphic.

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
\mathscr{BPreSh}(X_3; \mathscr{C}_3) & \xrightarrow{\pi^*[\mathscr{H}]} & \mathscr{BPreSh}(X_2; \mathscr{C}_2) & \xrightarrow{\pi^*[\mathscr{G}]} & \mathscr{BPreSh}(X_1; \mathscr{C}_1) \\
\uparrow \mathbf{y} & & \uparrow \mathbf{y} & & \uparrow \mathbf{y} \\
\mathscr{C}_3 & \xrightarrow[\mathscr{H}]{} & \mathscr{C}_2 & \xrightarrow[\mathscr{G}]{} & \mathscr{C}_1
\end{array}$$

The commutativity of the two inner squares implies the commutativity of the outer rectangle, so that $\pi^*[\mathscr{G} \circ \mathscr{H}] \cong_{\mathbf{h}\circ\mathbf{g}} \pi^*[\mathscr{G}] \circ \pi^*[\mathscr{H}]$, as was desired. ■

Theorem 2.5.10. In Theorem 2.5.7, if we assume also that $\mathscr{C}_+$ is X_+-finitely X_+-cocomplete and that $\mathscr{F}$ maps X_+-finite X_+-colimits to X_--colimits, then the $\mathfrak{f}$-functor $\pi^*[\mathscr{F}]: \mathscr{BPreSh}(X_+; \mathscr{C}_+) \to \mathscr{BPreSh}(X_-; \mathscr{C}_-)$ maps X_+-finite X_+-limits to X_--limits.

Proof. By FOTP the assumption that $\mathscr{C}_+$ is X_+-finitely X_+-complete implies that $\mathfrak{f}^*\mathscr{C}_+$ is X_--finitely X_--complete. By FOTP again the assumption that $\mathscr{F}$ maps X_+-finite X_+-colimits to X_--colimits implies that the X_--functor $\mathscr{F}_{X_-}: \mathfrak{f}^*\mathscr{C}_+ \to \mathscr{C}_-$ preserves X_--finite X_--colimits. Thus the Booleanization of Theorem 2.2.5 implies that the X_--functor $\pi^*[\mathscr{F}_{X_-}]: \mathfrak{f}^*\mathscr{C}_+ \to \mathscr{C}_-$ preserves X_--finite X_--limits. Since $\pi^*[\mathscr{F}] = \pi^*[\mathscr{F}_{X_-}] \circ \mathfrak{f}^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$ and $\mathfrak{f}^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$ maps X_+-finite X_+-limits to X_--limits by Theorem 2.5.5, the desired conclusion follows. ■

An $\mathfrak{f}$-left-exact $\mathfrak{f}$-functor $\mathscr{F}: \mathscr{C}_+ \to \mathscr{C}_-$ of small_1 $X_\pm$-categories $\mathscr{C}_\pm$ with $\mathscr{C}_+$ being X_+-finitely X_+-complete and $\mathscr{C}_\pm$ being endowed with Grothendieck

$X_\pm$-topologies $\mathscr{L}_\pm$ is called an f-*topological* f-*functor* from the X_+-site $(\mathscr{C}_+, \mathscr{L}_+)$ to the X_--site $(\mathscr{C}_-, \mathscr{L}_-)$. An f-topological f-functor $\mathscr{F}: (\mathscr{C}_+, \mathscr{L}_+) \to (\mathscr{C}_-, \mathscr{L}_-)$ is said to be f-*continuous* if for any $a \in \text{Ob}\, \mathscr{C}_+$ and any total $\mathscr{S} \in \mathscr{L}_+(a)$, the minimal $X_{-,E\mathscr{F}a}$-sieve on $\mathscr{F}a$ containing $\mathscr{F}\mathscr{S}$ belongs to $\mathscr{L}_-(\mathscr{F}a)$. Each f-topological f-functor $\mathscr{F}: (\mathscr{C}_+, \mathscr{L}_+) \to (\mathscr{C}_-, \mathscr{L}_-)$ gives rise to its associated f-functor

$$\pi^*[\mathscr{F}; \mathscr{L}_+, \mathscr{L}_-] \\ = a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ \imath_{\mathscr{L}_+}: \mathcal{BSh}(X_+; \mathscr{C}_+, \mathscr{L}_+) \to \mathcal{BSh}(X_-; \mathscr{C}_-, \mathscr{L}_-)$$

Let $\mathscr{F}: (\mathscr{C}_+, \mathscr{L}_+) \to (\mathscr{C}_-, \mathscr{L}_-)$ be an f-continuous f-topological f-functor. By FOTP it is easy to see the following.

Lemma 2.5.11. The X_--category $f^*\mathcal{BPreSh}(X_+; \mathscr{C}_+)$ can naturally be put down as an X_--subcategory of X_--category $\mathcal{BPreSh}(X_-; \mathscr{C}_-)$ with a natural injection

$$\jmath[\mathscr{C}_+]: f^*\mathcal{BPreSh}(X_+; \mathscr{C}_+) \to \mathcal{BPreSh}(X_-; \mathscr{C}_-)$$

and the following diagram is commutative up to natural X_--isomorphisms:

$$\begin{array}{ccc} \mathcal{BPreSh}(X_-; \mathscr{C}_-) & \xleftarrow{\quad \pi^*[\mathscr{F}_{X_-}] \quad} & \mathcal{BPreSh}(X_-; f^*\mathscr{C}_+) \\ & \nwarrow {\scriptstyle \pi^*[\mathscr{F}]_{X_-}} & \uparrow {\scriptstyle \jmath[\mathscr{C}_+]} \\ & & f^*\mathcal{BPreSh}(X_+; \mathscr{C}_+) \end{array}$$

We denote by $f^*\mathscr{L}_+$ the minimal Grothendieck X_--topology on the X_--category $f^*\mathscr{C}_+$ such that the f-functor consisting of the assignments $x \in \text{Ob}\, \mathscr{C}_+ \mapsto f^*x \in \text{Ob}\, f^*\mathscr{C}_+$ and $f \in \text{Mor}\, \mathscr{C}_+ \mapsto f^*f \in \text{Mor}\, f^*\mathscr{C}_+$ is an f-continuous f-topological f-functor from $(\mathscr{C}_+, \mathscr{L}_+)$ to $(f^*\mathscr{C}_+, f^*\mathscr{L}_+)$.

Lemma 2.5.12. The X_--category $f^*\mathcal{BSh}(X_+; \mathscr{C}_+, \mathscr{L}_+)$ can naturally be put down as an X_--subcategory of the X_--category $\mathcal{BSh}(X_-; f^*\mathscr{C}_+, f^*\mathscr{L}_+)$ with a natural injection

$$\jmath[\mathscr{C}_+, \mathscr{L}_+]: f^*\mathcal{BSh}(X_+; \mathscr{C}_+, \mathscr{L}_+) \to \mathcal{BSh}(X_-; f^*\mathscr{C}_+, f^*\mathscr{L}_+)$$

and the following diagram is commutative up to natural X_--isomorphisms:

$$\begin{array}{ccc}
\mathscr{BSh}(X_-;\, f^*\mathscr{C}_+,\, f^*\mathscr{L}_+) & \xleftarrow{\ \mathscr{a}_{f^*\mathscr{L}_+}\ } & \mathscr{BPreSh}(X_-;\, f^*\mathscr{C}_+) \\
\uparrow{\scriptstyle \jmath[\mathscr{C}_+,\, \mathscr{L}_+]} & & \uparrow{\scriptstyle \jmath[\mathscr{C}_+]} \\
f^*\mathscr{BSh}(X_+;\, \mathscr{C}_+,\, \mathscr{L}_+) & \xleftarrow[\ f^*\mathscr{a}_{\mathscr{L}_+}\ ]{} & f^*\mathscr{BPreSh}(X_+;\, \mathscr{C}_+)
\end{array}$$

Proof. We can use a Booleanized version of the colimit construction of Σ in Borceux (1994, Vol. 3, §3.3) for computing $\mathscr{a}_{\mathscr{L}_+}$ and $\mathscr{a}_{f^*\mathscr{L}_+}$. Thus the desired result follows readily from Theorem 2.5.5. ■

The proof of the above lemma shows also the following result.

Lemma 2.5.13. The following diagram is commutative up to natural X_--isomorphisms:

$$\begin{array}{ccc}
\mathscr{BPreSh}(X_-;\, f^*\mathscr{C}_+) & \xleftarrow{\ \imath_{f^*\mathscr{L}_+}\ } & \mathscr{BSh}(X_-;\, f^*\mathscr{C}_+,\, f^*\mathscr{L}_+) \\
\uparrow{\scriptstyle \jmath[\mathscr{C}_+]} & & \uparrow{\scriptstyle \jmath[\mathscr{C}_+,\, \mathscr{L}_+]} \\
f^*\mathscr{BPreSh}(X_+;\, \mathscr{C}_+) & \xleftarrow[\ f^*\imath_{\mathscr{L}_+}\ ]{} & f^*\mathscr{BSh}(X_+;\, \mathscr{C}_+,\, \mathscr{L}_+)
\end{array}$$

Theorem 2.5.14. The f-functors $\mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]$ and $\mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ \imath_{\mathscr{L}_+} \circ \mathscr{a}_{\mathscr{L}_+}$ from $\mathscr{BSh}(X_+;\, \mathscr{C}_+,\, \mathscr{L}_+)$ to $\mathscr{BSh}(X_-;\, \mathscr{C}_-,\, \mathscr{L}_-)$ are naturally f-isomorphic.

Proof. On the basis of Theorem 2.5.2 it suffices to show that X_--functors $(\mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}])_{X_-}$ and $(\mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ \imath_{\mathscr{L}_+} \circ \mathscr{a}_{\mathscr{L}_+})_{X_-}$ are naturally X_--isomorphic. On the basis of Lemma 2.5.3 we have $(\mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}])_{X_-} = \mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}])_{X_-}$ and

$$\begin{aligned}
(\mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ \imath_{\mathscr{L}_+} \circ \mathscr{a}_{\mathscr{L}_+})_{X_-} &= \mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]_{X_-} \circ f^*(\imath_{\mathscr{L}_+} \circ \mathscr{a}_{\mathscr{L}_+}) \\
&= \mathscr{a}_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]_{X_-} \circ f^*\imath_{\mathscr{L}_+} \circ f^*\mathscr{a}_{\mathscr{L}_+}
\end{aligned}$$

Since $\mathcal{F}_{X_-}$: $f^*\mathcal{C}_+ \to \mathcal{C}_-$ is X_--left-exact with $f^*\mathcal{C}_+$ being X_--finitely X_--complete,

$\pi[\mathcal{F}_{X_-}]$

$$= (\pi_*[\mathcal{F}_{X_-}], \pi^*[\mathcal{F}_{X_-}]): \mathcal{BPreSh}(X_-; \mathcal{C}_-) \to \mathcal{BPreSh}(X_-; f^*\mathcal{C}_+)$$

is an X_--geometric morphism. Therefore the Booleanization of Theorem 2.1.7 guarantees that X_--functors $a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}_{X_-}] \circ i_{f^*\mathcal{L}_+} \circ a_{f^*\mathcal{L}_+}$ and $a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}_{X_-}]$ are naturally X_--isomorphic. Therefore we have

$a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}]_{X_-} \circ f^* i_{\mathcal{L}_+} \circ f^* a_{\mathcal{L}_+}$

$\cong_{X_-} a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}_{X_-}] \circ \jmath[\mathcal{C}_+] \circ f^* i_{\mathcal{L}_+} \circ f^* a_{\mathcal{L}_+}$ (Lemma 2.5.11)

$\cong_{X_-} a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}_{X_-}] \circ i_{f^*\mathcal{L}_+} \circ \jmath[\mathcal{C}_+, \mathcal{L}_+] \circ f^* a_{\mathcal{L}_+}$ (Lemma 2.5.13)

$\cong_{X_-} a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}_{X_-}] \circ i_{f^*\mathcal{L}_+} \circ a_{f^*\mathcal{L}_+} \circ \jmath[\mathcal{C}_+]$ (Lemma 2.5.12)

$\cong_{X_-} a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}_{X_-}] \circ \jmath[\mathcal{C}_+]$

(Booleanization of Theorem 2.1.7)

$\cong_{X_-} a_{\mathcal{L}_-} \circ \pi^*[\mathcal{F}]_{X_-}$ (Lemma 2.5.11)

Thus the desired result follows at once. ■

Theorem 2.5.15. Let g: $X_1 \to X_2$ and h: $X_2 \to X_3$ be morphisms of **BLoc**. Let $\mathcal{G}$: $(\mathcal{C}_2, \mathcal{L}_2) \to (\mathcal{C}_1, \mathcal{L}_1)$ be a g-continuous g-topological g-functor and $\mathcal{H}$: $(\mathcal{C}_3, \mathcal{L}_3) \to (\mathcal{C}_2, \mathcal{L}_2)$ an h-continuous h-topological h-functor. Then the h ∘ g-functors $\pi^*[\mathcal{G} \circ \mathcal{H}; \mathcal{L}_3, \mathcal{L}_1)$ and $\pi^*[\mathcal{G}; \mathcal{L}_2, \mathcal{L}_1] \circ \pi^*[\mathcal{H}; \mathcal{L}_3, \mathcal{L}_2]$ from $\mathcal{BSh}(X_3; \mathcal{C}_3, \mathcal{L}_3)$ to $\mathcal{BSh}(X_1; \mathcal{C}_1, \mathcal{L}_1)$ are naturally h ∘ g-isomorphic.

Proof. It suffices to note that

$\pi^*[\mathcal{G}; \mathcal{L}_2, \mathcal{L}_1] \circ \pi^*[\mathcal{H}; \mathcal{L}_3, \mathcal{L}_2]$

$= a_{\mathcal{L}_1} \circ \pi^*[\mathcal{G}] \circ i_{\mathcal{L}_2} \circ a_{\mathcal{L}_2} \circ \pi^*[\mathcal{H}] \circ i_{\mathcal{L}_3}$

$\cong_{h \circ g} a_{\mathcal{L}_1} \circ \pi^*[\mathcal{G}] \circ \pi^*[\mathcal{H}] \circ i_{\mathcal{L}_3}$ (Theorem 2.5.14)

$\cong_{h \circ g} a_{\mathcal{L}_1} \circ \pi^*[\mathcal{G} \circ \mathcal{H}] \circ i_{\mathcal{L}_3}$ (Theorem 2.5.9)

$= \pi^*[\mathcal{G} \circ \mathcal{H}; \mathcal{L}_3, \mathcal{L}_1]$ ■

3. EMPIRICAL GROTHENDIECK TOPOSES

Let $\mathfrak{M}$ be a manual of Boolean locales, which shall be fixed throughout this section. We assume that the reader is appreciably familiar with Section 3 of our previous paper (Nishimura, 1995c). In particular, he or she should feel at home with such a locution as an "empirical framework over $\mathfrak{M}$."

3.1. Empirical Algebraic Theories

The assignments $X \in \mathrm{Ob}\,\mathfrak{M} \mapsto (X, \mathcal{BAT\!h}(X))$ and $f \in \mathrm{Mor}\,\mathfrak{M} \mapsto (f, f^*_{\mathcal{BAT\!h}})$ constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{EAIh}[\mathfrak{M}]$ or simply by $\mathfrak{EAIh}$. The objects of **EObj**($\mathfrak{EAIh}$) are called *empirical algebraic theories over* $\mathfrak{M}$.

Let $\mathfrak{I}$ be an empirical algebraic theory over $\mathfrak{M}$. The assignments

$$X \in \mathrm{Ob}\,\mathfrak{M} \mapsto (X, \mathcal{BACat}(X; \mathfrak{I}_{\mathcal{O}b}(X)))$$

and

$$f: X \to Y \in \mathrm{Mor}\,\mathfrak{M} \mapsto (f, f^*_{\mathcal{BACat}}[\mathfrak{I}_{\mathcal{O}b}(X), \mathfrak{I}_{\mathcal{O}b}(Y), \mathfrak{I}_{\mathcal{O}b}(f)])$$

constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{EACat}[\mathfrak{M}; \mathfrak{I}]$ or simply by $\mathfrak{EACat}[\mathfrak{I}]$. The objects of the category **EObj**($\mathfrak{EACat}[\mathfrak{I}]$) are called *empirical* $\mathfrak{I}$*-algebras over* $\mathfrak{M}$. The assignments

$$X \in \mathrm{Ob}\,\mathfrak{M} \mapsto (X, \mathcal{BACat}_{\mathrm{fg}}(X; \mathfrak{I}_{\mathcal{O}b}(X)))$$

and

$$f: X \to \in Y\ \mathrm{Mor}\,\mathfrak{M} \mapsto (f, f^*_{\mathcal{BACat}_{\mathrm{fg}}}[\mathfrak{I}_{\mathcal{O}b}(X), \mathfrak{I}_{\mathcal{O}b}(Y), \mathfrak{I}_{\mathcal{O}b}(f)])$$

constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{EACat}_{\mathrm{fg}}[\mathfrak{M}; \mathfrak{I}]$ or simply by $\mathfrak{EACat}_{\mathrm{fg}}[\mathfrak{I}]$.

Example 3.1.1. Since each $f: X \to Y \in \mathrm{Mor}\,\mathfrak{M}$ naturally enables us to regard $f^*\mathcal{T}_{\mathcal{BR}ng}(Y)$ as an X-subcategory of $\mathcal{T}_{\mathcal{BR}ng}(X)$, the assignment $X \in \mathrm{Ob}\,\mathfrak{M} \mapsto (X, \mathcal{BAT\!h}(X), \mathcal{T}_{\mathcal{BR}ng}(X))$ gives rise to an empirical algebraic theory over $\mathfrak{M}$ to be denoted by $\mathfrak{I}_{\mathfrak{ERng}}[\mathfrak{M}]$ and to be called an *empirical ring theory over* $\mathfrak{M}$. An object of **EObj**($\mathfrak{EACat}[\mathfrak{I}_{\mathfrak{ERng}}[\mathfrak{M}]]$) is called an *empirical ring over* $\mathfrak{M}$. ■

Example 3.1.2. Let $\mathfrak{R}$ be an empirical ring over $\mathfrak{M}$. Since each $f: X \to Y \in \mathrm{Mor}\,\mathfrak{M}$ naturally gives rise to a morphism from $f^*\mathcal{T}_{\mathfrak{R}Y-\mathcal{BA}lg}(Y)$ to $\mathcal{T}_{\mathfrak{R}X-\mathcal{BA}lg}(X)$ in the X-category $\mathcal{BAT\!h}(X)$, the assignment $X \in \mathrm{Ob}\,\mathfrak{M} \mapsto (X, \mathcal{BAT\!h}(X), \mathcal{T}_{\mathfrak{R}X-\mathcal{BA}lg}(X))$ gives rise to an empirical algebraic theory over $\mathfrak{M}$ to be denoted by $\mathfrak{I}_{\mathfrak{R}-\mathfrak{Alg}}[\mathfrak{M}]$ and to be called an *empirical theory of* $\mathfrak{R}$-

algebras over $\mathfrak{M}$. An object of $\mathbf{EObj}(\mathfrak{EACat}[\mathfrak{I}_{\mathfrak{R}\text{-}\mathfrak{Alg}}[\mathfrak{M}]])$ is called an *empirical* $\mathfrak{R}$-algebra over $\mathfrak{M}$.

Example 3.1.3. Since each $f: X \to Y \in \text{Mor}\, \mathfrak{M}$ naturally gives rise to a morphism from $f^*\mathscr{T}_{\mathscr{BSAlg}}(Y)$ to $\mathscr{T}_{\mathscr{BSAlg}}(X)$ in the X-category $\mathscr{BAFK}(X)$, the assignment $X \in \text{Ob}\, \mathfrak{M} \mapsto (X, \mathscr{BAFK}(X), \mathscr{T}_{\mathscr{BSAlg}}(X))$ gives rise to an empirical algebraic theory over $\mathfrak{M}$ to be denoted by $\mathfrak{I}_{\mathscr{BS}\mathfrak{Alg}}[\mathfrak{M}]$ and to be called an *empirical theory of smooth algebras over* $\mathfrak{M}$. An object of $\mathbf{EObj}(\mathfrak{EACat}[\mathfrak{I}_{\mathscr{BS}\mathfrak{Alg}}[\mathfrak{M}]])$ is called an *empirical smooth algebra over* $\mathfrak{M}$.

3.2. Empirical Grothendieck Toposes

An $\mathfrak{M}$-*site* is a pair (Φ, ι) of an empirical framework Φ over $\mathfrak{M}$ and an assignment ι to each $X \in \text{Ob}\, \mathfrak{M}$ of a Grothendieck topology $\iota(X)$ on the X-category $\Phi_{\mathscr{Cat}}(X)$ yielding the following conditions:

(3.2.1) For each $X \in \text{Ob}\, \mathfrak{M}$, the X-category $\Phi_{\mathscr{Cat}}(X)$ is X-finitely X-complete.

(3.2.2) For each $f: X \to Y \in \text{Mor}\, \mathfrak{M}$, the f-functor $\Phi_{\mathscr{Cat}}(f): \Phi_{\mathscr{Cat}}(Y) \to \Phi_{\mathscr{Cat}}(X)$ is f-left-exact.

(3.2.3) For each $f \in \text{Mor}\, \mathfrak{M}$, the f-topological f-functor $\Phi_{\mathscr{Cat}}(f)$: $(\Phi_{\mathscr{Cat}}(Y), \iota(Y)) \to (\Phi_{\mathscr{Cat}}(X), \iota(X))$ is f-continuous.

Given an $\mathfrak{M}$-site (Φ, ι), the *Grothendieck* $\mathfrak{M}$-*topos associated with* (Φ, ι) is an empirical framework $\mathfrak{ESh}(\mathfrak{M}; \Phi, \iota)$ over $\mathfrak{M}$ with $\Phi_{\mathscr{Cat}}$ assigning to each $X \in \text{Ob}\, \mathfrak{M}$ of the X-category $\mathscr{BSh}(X; \Phi X, \iota(X))$ and to each $f: X \to Y \in \text{Mor}\, \mathfrak{M}$ of the f-functor

$$\pi^*[\Phi f; \iota(Y), \iota(X)]: \mathscr{BSh}(Y; \Phi Y, \iota(Y)) \to \mathscr{BSh}(X; \Phi X, \iota(X))$$

The objects of $\mathbf{EObj}(\mathfrak{ESh}(\mathfrak{M}; \Phi, \iota))$ are called *empirical sheaves over* (Φ, ι).

Example 3.2.1. Let $\mathfrak{R}$ be an empirical ring over $\mathfrak{M}$, we denote by $\iota_{\mathfrak{R}\text{-}\mathfrak{EZar}}$ the assignment

$$X \in \text{Ob Mor}\, \mathfrak{M} \mapsto \mathscr{L}_{\mathfrak{R}_{\mathscr{Ob}}(X)\text{-}\mathscr{BZar}}[X]$$

It is easy to see that the pair $(\mathfrak{EACat}_{fg}(\mathfrak{M}; \mathfrak{I}_{\mathfrak{R}\text{-}\mathfrak{EAlg}})^{op}, \iota_{\mathfrak{R}\text{-}\mathfrak{EZar}})$ is an $\mathfrak{M}$-site. Its resulting Grothendieck $\mathfrak{M}$-topos

$$\mathfrak{ESh}(\mathfrak{M}; \mathfrak{EACat}_{fg}(\mathfrak{M}; \mathfrak{I}_{\mathfrak{R}\text{-}\mathfrak{EAlg}})^{op}, \iota_{\mathfrak{R}\text{-}\mathfrak{EZar}})$$

is called the *Zariski* $\mathfrak{M}$-*topos over* $\mathfrak{R}$.

Example 3.2.2. We denote by $\iota_{\mathfrak{E}\mathscr{S}\mathfrak{Zar}}$ the assignment $X \in \text{Ob}\, \mathfrak{M} \mapsto \mathscr{L}_{\mathscr{BSZar}}[X]$. It is easy to see that the pair $(\mathfrak{EACat}_{fg}(\mathfrak{M}; \mathfrak{I}_{\mathfrak{E}\mathscr{S}\mathfrak{Alg}})^{op}, \iota_{\mathfrak{E}\mathscr{S}\mathfrak{Zar}})$ is an $\mathfrak{M}$-site. Its resulting Grothendieck $\mathfrak{M}$-topos

$$\mathfrak{ESh}(\mathfrak{M};\ \mathfrak{EACat}_{fg}(\mathfrak{M};\ \mathfrak{I}_{\mathfrak{E}\mathcal{S}\mathfrak{Alg}})^{op},\ \iota_{\mathfrak{E}\mathcal{S}\mathfrak{Zar}})$$

is called the *smooth Zariski* $\mathfrak{M}$*-topos*. ■

APPENDIX: MISCELLANEOUS EXAMPLES

Let k be an arbitrary small$_0$ ring, which shall be fixed throughout the succeeding two examples. We first give an example of an algebraic theory in Lawvere's (1963) form. For treatments of algebraic theories à la Lawvere (1963) the reader is referred to Borceux (1994, Vol. 2, §3) and Schubert (1972, §18) as well as Lawvere's (1963) epoch-making dissertation.

Example A.1. The algebraic theory $T_{\mathbf{Rng}}$ has N as its set of objects. Given $n, m \in \mathbf{N}$, its morphisms from n to m are all m-tuples of polynomial functions of n variables with coefficients in $\mathbf{Z}$. Note that each $n \in \mathbf{N}$ is an nth power of 1 in the category $T_{\mathbf{Rng}}$. The algebraic category $\mathbf{ACat}(T_{\mathbf{Rng}})$ corresponding to $T_{\mathbf{Rng}}$ consists of all finite-products-preserving functors from $T_{\mathbf{Rng}}$ to Ens_0 and natural transformations among them. It is equivalent to the category of small$_0$ rings and homomorphisms of rings in the standard sense, so that these two categories are naively identified unless confusion may arise.

A small$_0$ ring k shall be fixed for the following two examples.

Example A.2. The algebraic theory $T_{k\text{-}\mathbf{Alg}}$ has $\mathbf{N}$ as its set of objects. Given $n, m \in \mathbf{N}$, its morphisms from n to m are all m-tuples of polynomial functions of n variables with coefficients in k. Note that each $n \in \mathbf{N}$ is an nth power of 1 in the category $T_{k\text{-}\mathbf{Alg}}$. The algebraic category $\mathbf{ACat}(T_{k\text{-}\mathbf{Alg}})$ corresponding to $T_{k\text{-}\mathbf{Alg}}$ consists of all finite-products-preserving functors from $T_{k\text{-}\mathbf{Alg}}$ to $\mathbf{Ens}_0$ and natural transformations among them. It is equivalent to the category of small$_0$ k-algebras and homomorphisms of k-algebras in the standard sense, so that these two categories are naively identified unless confusion may arise. The full subcategory of $\mathbf{ACat}(T_{k\text{-}\mathbf{Alg}})$ whose objects are all finitely generated k-algebras is denoted by $\mathbf{ACat}_{fg}(T_{k\text{-}\mathbf{Alg}})$. If an object A of $\mathbf{ACat}_{fg}(T_{k\text{-}\mathbf{Alg}})$ is to be regarded as an object of $\mathbf{ACat}_{fg}(T_{k\text{-}\mathbf{Alg}})^{op}$, then it is denoted by $\ell(A)$. Similarly, if a morphism $f\colon A \to B$ of $\mathbf{ACat}_{fg}(T_{k\text{-}\mathbf{Alg}})$ is to be regarded as a morphism from $\ell(B)$ to $\ell(A)$ of the category $\mathbf{ACat}_{fg}(T_{k\text{-}\mathbf{Alg}})^{op}$, it is denoted by $\ell(f)$.

Example A.3. The minimal Grothendieck topology $L_{k\text{-}\mathbf{Zar}}$ over $\mathbf{ACat}_{fg}(T_{k\text{-}\mathbf{Alg}})^{op}$ such that for any finitely generated k-algebra A and any finite elements $a_1, \ldots, a_n$ of A with $1 \in (a_1, \ldots, a_n)$ the family $\{\ell(p_i)\colon \ell(A[a_i^{-1}]) \to \ell(A)\}_{i=1}^{n}$ with $p_i\colon A \to A[a_i^{-1}]$ being the canonical projection for each i $(1 \leq i \leq n)$ $L_{k\text{-}\mathbf{Zar}}$-covers $\ell(A)$ is called the *Zariski topology over k*. The Grothendieck topos $\mathbf{E}_{k\text{-}\mathbf{Zar}}$ associated with the site $(\mathbf{ACat}_{fg}(T_{k\text{-}\mathbf{Alg}})^{op}, L_{k\text{-}\mathbf{Zar}})$ is called the *Zariski topos over k.*

Example A.4. The algebraic theory $T_{\mathbf{SAlg}}$ has $\mathbf{R}^n$'s ($n \in \mathbf{N}$) as its objects. The morphisms from $\mathbf{R}^n$ to $\mathbf{R}^m$ are all smooth functions from $\mathbf{R}^n$ to $\mathbf{R}^m$. The objects of the algebraic category $\mathbf{ACat}(T_{\mathbf{SAlg}})$ corresponding to $T_{\mathbf{SAlg}}$ are all finite-products-preserving functors from $T_{\mathbf{SAlg}}$ to $\mathbf{Ens}_0$ and are called *smooth algebras*, which Moerdijk and Reyes (1991) called *C^∞-rings*. The morphisms of $\mathbf{ACat}(T_{\mathbf{SAlg}})$ are simply all natural transformations among them. The full subcategory $\mathbf{ACat}_{\mathrm{fg}}(T_{\mathbf{SAlg}})$ of $\mathbf{ACat}_{\mathrm{fg}}(T_{\mathbf{SAlg}})$ and an operation ℓ: $\mathbf{ACat}_{\mathrm{fg}}(T_{\mathbf{SAlg}}) \to \mathbf{ACat}_{\mathrm{fg}}(T_{\mathbf{SAlg}})^{\mathrm{op}}$ are defined as in Example A.2.

Example A.5. The *smooth Zariski topology* $L_{\mathbf{SZar}}$ is defined as in Example A.3 to be a Grothendieck topology over the category $\mathbf{ACat}_{\mathrm{fg}}(T_{\mathbf{SAlg}})^{\mathrm{op}}$. The Grothendieck topos $\mathbf{E}_{\mathbf{SZar}}$ associated with the site $(\mathbf{ACat}_{\mathrm{fg}}(T_{\mathbf{SAlg}})^{\mathrm{op}}, L_{\mathbf{SZar}})$ is called the *smooth Zariski topos.*

Let X be a Boolean locale, which shall be fixed throughout the remainder of this section. Therefore Booleanization shall always mean Booleanization with respect to X.

Example A.6. The notion of the algebraic theory $T_{\mathbf{Rng}}$ in Example A.1 can be Booleanized, and we get the notion of the algebraic X-theory $\mathcal{T}_{\mathcal{BR}ng}(\mathrm{X})$. The total objects of $\mathcal{BACat}(\mathrm{X}; \mathcal{T}_{\mathcal{BR}ng}(\mathrm{X}))$ are called X-*rings*.

Let $\mathcal{R}$ be an X-ring, which shall be fixed for the following two examples.

Example A.7. The notion of the algebraic theory $T_{k\text{-}\mathbf{Alg}}$ in Example A.2 can be Booleanized, and we get the notion of the algebraic X-theory $\mathcal{T}_{\mathcal{R}\text{-}\mathcal{BA}lg}(\mathrm{X})$. The total objects of $\mathcal{BACat}(\mathrm{X}; \mathcal{T}_{\mathcal{R}\text{-}\mathcal{BA}lg}(\mathrm{X}))$ are called X-*algebras over* $\mathcal{R}$.

Example A.8. The notion of the Grothendieck topology $L_{k\text{-}\mathbf{Zar}}$ in Example A.3 can be Booleanized, and we get the notion of the Grothendieck X-topology $\mathcal{L}_{\mathcal{R}\text{-}\mathcal{BZ}ar}[\mathrm{X}]$ over the X-category $\mathcal{BACat}_{\mathrm{fg}}(\mathrm{X}; \mathcal{T}_{\mathcal{R}\text{-}\mathcal{BA}lg}[\mathrm{X}])$.

Example A.9. The notion of the algebraic theory $T_{\mathbf{SAlg}}$ in Example A.4 can be Booleanized, and we get the notion of the algebraic X-theory $\mathcal{T}_{\mathcal{BSA}lg}[\mathrm{X}]$. The total objects of $\mathcal{BACat}(\mathrm{X}; \mathcal{T}_{\mathcal{R}\text{-}\mathcal{BA}lg}[\mathrm{X}])$ are called *smooth* X-*algebras.*

Example A.10. The notion of the Grothendieck topology $L_{\mathbf{SZar}}$ in Example A.5 can be Booleanized, and we get the notion of the Grothendieck X-topology $\mathcal{L}_{\mathcal{BSZ}ar}[\mathrm{X}]$ over the X-category $\mathcal{BACat}_{\mathrm{fg}}(\mathrm{X}; \mathcal{T}_{\mathcal{BSA}lg}[\mathrm{X}])$.

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